

# ASSOCIATED GRADED OF HODGE MODULES AND CATEGORICAL $\mathfrak{sl}_2$ ACTIONS

SABIN CAUTIS, CHRISTOPHER DODD, AND JOEL KAMNITZER

**ABSTRACT.** One of the most mysterious aspects of Saito's theory of Hodge modules are the Hodge and weight filtrations that accompany the pushforward of a Hodge module under an open embedding. In this paper we consider the open embedding in a product of complementary Grassmannians given by pairs of transverse subspaces. The push-forward of the structure sheaf under this open embedding is an important Hodge module from the viewpoint of geometric representation theory and homological knot invariants.

We compute the associated graded of this push-forward with respect to the induced Hodge filtration as well as the resulting weight filtration. The main tool is a categorical  $\mathfrak{sl}_2$  action (which is of independent interest) on the category of  $\mathcal{D}_h$ -modules on Grassmannians.

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## 1. INTRODUCTION

**1.1. Mixed Hodge modules and their associated graded.** The theory of mixed Hodge modules was developed by Saito as a  $\mathcal{D}$ -module analog of the theory of weights for  $\ell$ -adic sheaves. Let  $X$  be a smooth proper algebraic variety. A mixed Hodge module  $M$  on  $X$  gives rise to filtered  $\mathcal{D}_X$ -module  $\mathbf{G}(M)$  and thus to a  $\mathcal{O}_{T^*X}$ -module  $\mathrm{gr}(\mathbf{G}(M))$ .

The most non-trivial aspect of Saito's theory concerns push-forward of mixed Hodge modules along non-proper maps. For example, suppose that  $j : U \hookrightarrow X$  is the inclusion of an open subset and consider the mixed Hodge module push-forward  $j_*\mathcal{O}_U$ . This results in a subtle filtered structure on  $j_*\mathcal{O}_U$  which reflects the singularities of the complement  $X \setminus U$ . More precisely,  $j_*\mathcal{O}_U$  has a simple description if the complement is simple normal crossing. For

more complicated complements one has to blow up along  $X \setminus U$  to obtain a simple normal crossing complement and then compute the proper pushforward of  $j_*\mathcal{O}_U$ . In general this is quite difficult to do in practice. For a more general discussion see, for instance, [Sc].

In this paper we compute  $\mathrm{gr}(\mathbb{G}(j_*\mathcal{O}_U))$  for a natural open embedding  $j$  that shows up in geometric representation theory and has connections to other areas, including higher representation theory and homological knot invariants. More specifically, we take  $X = \mathbb{G}(k, N) \times \mathbb{G}(N - k, N)$  the product of two complementary Grassmannians and take  $U$  to be the locus of pairs of subspaces which intersect trivially. Our main result (Theorem 8.4 and Corollary 9.5) identifies  $\mathrm{gr}(\mathbb{G}(j_*\mathcal{O}_U))$  as the pushforward of a line bundle of a certain locally closed subset of  $T^*X$ . We also show that the resulting weight filtration on  $\mathrm{gr}(\mathbb{G}(j_*\mathcal{O}_U))$  is a natural filtration from the coherent sheaves viewpoint, related to the different components of  $T^*\mathbb{G}(k, N) \times_{\mathrm{gl}_N} T^*\mathbb{G}(N - k, N)$ . This computation is quite indirect and uses the theory of categorical  $\mathfrak{sl}_2$  actions.

**1.2.  $\mathfrak{sl}_2$ -actions involving Grassmannians.** Consider the quantum group  $U_q(\mathfrak{sl}_2)$  with its usual generators  $E, F, K$ . Going back to work of Beilinson-Lusztig-MacPherson [BLM], there is a geometric incarnation of  $U_q(\mathfrak{sl}_2)$  involving Grassmannians  $\mathbb{G}(k, N)$  over a finite field with  $q^2$  elements. In this setup,  $E$  acts on a subspace  $W$  to produce the formal sum of all hyperplanes of  $W$  and  $F$  acts on  $W$  to produce the formal sum of all subspaces in which  $W$  sits as a hyperplane. This defines a representation of  $U_q(\mathfrak{sl}_2)$  on the vector space  $\bigoplus_{k=0}^N \mathbb{C}[\mathbb{G}(k, N)]$ .

Using Grothendieck's faisceaux-fonctions correspondence, there is a natural categorification of this construction. Consider the categories  $D_c(\mathbb{G}(k, N))$  (now using Grassmannians over  $\mathbb{C}$ ) of constructible sheaves. The natural incidence correspondence  $C(k, N) \subset \mathbb{G}(k, N) \times \mathbb{G}(k - 1, N)$  can be used to define a functor  $\mathbf{E} : D_c(\mathbb{G}(k, N)) \rightarrow D_c(\mathbb{G}(k - 1, N))$  and a similar functor  $\mathbf{F}$  in the opposite direction. It is not difficult to prove that these functors satisfy the defining relations of  $U_q(\mathfrak{sl}_2)$ . This idea seems to have been known to experts for some time and first appeared in the literature in the work of Zheng (see Theorem 3.3.6 of [Z1]). By the Riemann-Hilbert correspondence this gives a categorical  $\mathfrak{sl}_2$  action on  $\mathcal{D}$ -modules on Grassmannians.

**1.3. Cotangent bundles to Grassmannians and filtered  $\mathcal{D}$ -modules.** On the other hand, in [CKL1], we constructed a categorical  $\mathfrak{sl}_2$ -action on the categories of coherent sheaves on cotangent bundles to Grassmannians. In other words, we defined kernels

$$\mathcal{E} \in D(\mathcal{O}_{T^*\mathbb{G}(k, N) \times T^*\mathbb{G}(k-1, N)}) \quad \text{and} \quad \mathcal{F} \in D(\mathcal{O}_{T^*\mathbb{G}(k-1, N) \times T^*\mathbb{G}(k, N)})$$

using the conormal bundle of  $C(k, N)$  and proved that they induce a categorical  $\mathfrak{sl}_2$ -action.

After learning about our work, Roman Bezrukavnikov suggested to us (back in 2008) that we should be able to relate our construction from [CKL1] with Zheng's construction using the machinery of filtered  $\mathcal{D}$ -modules and Saito's theory of mixed Hodge modules. This is one of the things we carry out in this paper.

More precisely, we work with sheaves of  $\mathcal{D}_{X, h}$ -modules on smooth varieties  $X$ , where  $\mathcal{D}_{X, h}$  is the Rees algebra of differential operators on  $X$ . This is a sheaf of algebras over  $\mathbb{C}[h]$  and can be specialized at  $h = 0$  to obtain  $\pi_*\mathcal{O}_{T^*X}$ , the push-forward of the structure sheaf of the cotangent bundle. This gives us an “associated graded” functor  $\mathrm{gr} : \mathcal{D}_{X, h}\text{-mod} \rightarrow \mathcal{O}_{T^*X}\text{-mod}$ . We study some general properties of this functor in section 3 and in the context of kernels in section 4. In section 5, we review some results from the theory of mixed Hodge modules.

Working with Hodge modules is useful because we have results, such as the base change theorem, which do not hold for  $\mathcal{D}_h$ -modules. On the other hand, non-flat pullback (and in particular the tensor product) of Hodge modules is not compatible with the functor  $G$  from Hodge modules to  $\mathcal{D}_h$ -modules. In particular, this means that the kernel formalism for Hodge modules is not compatible with  $G$ . For this reason we need to consider both Hodge modules and  $\mathcal{D}_h$ -modules in this paper.

In section 6 we define a categorical  $\mathfrak{sl}_2$ -action on the category of  $\mathcal{D}_h$ -modules on products of Grassmannians. The kernels for this action are the  $\mathcal{D}_h$ -module versions of Zheng's construction (after applying the Riemann-Hilbert correspondence). In section 7 we prove that taking the associated graded of this action recovers our action from [CKL1] (up to conjugating by some line bundles).

**1.4. The associated graded of the equivalence.** One of the main applications of categorical  $\mathfrak{sl}_2$  actions (and the reason we studied them in [CKL1]) is that they can be used to construct equivalences. More precisely, given a categorical  $\mathfrak{sl}_2$ -action on some categories  $\mathcal{D}(\lambda)$ ,  $\lambda \in \mathbb{Z}$  we obtain an equivalence of categories  $T : \mathcal{D}(\lambda) \rightarrow \mathcal{D}(-\lambda)$ , which categorifies the quantum Weyl group element of  $U_q(\mathfrak{sl}_2)$ .  $T$  is defined as the iterated cone of the “Rickard complex”

$$F^{(\lambda)} \rightarrow F^{(\lambda+1)}E \rightarrow F^{(\lambda+2)}E^{(2)} \rightarrow \dots$$

If one applies this construction to the categorical  $\mathfrak{sl}_2$  actions discussed above we obtain equivalences

$$D(\mathcal{D}_{\mathbb{G}(k,N)}\text{-mod}) \xrightarrow{T} D(\mathcal{D}_{\mathbb{G}(N-k,N)}\text{-mod}) \quad \text{and} \quad D(\mathcal{O}_{T^*\mathbb{G}(k,N)}\text{-mod}) \xrightarrow{T'} D(\mathcal{O}_{T^*\mathbb{G}(N-k,N)}\text{-mod}).$$

Remarkably, both these equivalences have simpler and more explicit descriptions.

On the  $\mathcal{D}$ -module side the equivalence  $T$  is given by the kernel  $j_*\mathcal{O}_U$  where  $U \subset \mathbb{G}(k, N) \times \mathbb{G}(N-k, N)$  is the open locus discussed above, namely pairs  $(V, V')$  with  $V \cap V' = 0$ , and  $j$  is its embedding. This result first appeared (without proof) in [CR]. In the current paper we give a proof and extend the result to the context of  $\mathcal{D}_h$ -modules (see Theorem 8.4). In particular, we show that  $G(j_*\mathcal{O}_U)$  is the  $\mathcal{D}_h$ -module kernel that induces  $T$  and that the associated weight filtration on  $G(j_*\mathcal{O}_U)$  is the same as the natural one coming from the Rickard complex defining  $T$ . This result is related to a similar result (in the context of  $\ell$ -adic sheaves) due to Webster-Williamson [WW].

In [C1] we studied the kernel for  $T'$  in the category of coherent sheaves. We proved that it coincides with the (underived) push-forward  $R^0f_*(\mathcal{L})$ , where  $f$  is the inclusion of a dense, open subset  $\mathfrak{Z}^o \subset T^*\mathbb{G}(k, N) \times_{\mathfrak{sl}_N} T^*\mathbb{G}(N-k, N)$  and  $\mathcal{L}$  is an explicit line bundle.

Thus, as a consequence of our results, we prove that there is an isomorphism  $\text{gr}(G(j_*\mathcal{O}_U)) \cong R^0f_*(\mathcal{L})$  (Corollary 9.5). This is a purely geometric result, which (as far as we know) can only be deduced using this machinery of categorical  $\mathfrak{sl}_2$ -actions.

**1.5. Generalizations.** There are two possible directions for generalizing this work. In one direction, which will hopefully appear in a future paper [CDK], we replace  $\mathfrak{sl}_2$  by any simply-laced simple Lie algebra  $\mathfrak{g}$  and replace  $T^*\mathbb{G}(k, N)$  with the corresponding Nakajima quiver varieties. In [CKL2] we defined a categorical  $\mathfrak{g}$ -action using coherent sheaves on these quiver varieties. On the other hand, Webster [W] (building on Zheng [Z2]), defined a categorical  $\mathfrak{g}$ -action on a quotient category of the category of equivariant  $\mathcal{D}$ -modules on an affine space related to the quiver. The results of his paper can be generalized by working with equivariant

$\mathcal{D}_h$  modules. Again, one can relate the two constructions using the associated graded functor. In this setting, the equivalences  $\mathsf{T}$  and  $\mathsf{T}'$  are replaced by braid group actions [CK3] (of type  $\mathfrak{g}$ ).

On the other hand, we can try to replace  $\mathbb{G}(k, N)$  by any arbitrary co-minuscule flag variety  $G/P$ . In this context we do not expect categorical actions of any Lie algebra. However, we do expect similar equivalences

$$D(\mathcal{D}_{G/P\text{-mod}}) \xrightarrow{\mathsf{T}} D(\mathcal{D}_{G/Q\text{-mod}}) \quad \text{and} \quad D(\mathcal{O}_{T^*G/P\text{-mod}}) \xrightarrow{\mathsf{T}'} D(\mathcal{O}_{T^*G/Q\text{-mod}})$$

where  $G/Q$  denotes the opposite flag variety. We expect that  $\mathsf{T}, \mathsf{T}'$  are both given by natural complexes and by an “open push-forward” description, similar to the case of  $\mathbb{G}(k, N)$ . In section 10 we propose some precise conjectures along these lines. These equivalences for cominuscule flag varieties are of particular interest to us since they fit into our program for constructing knot homologies [CK2] (that program has only been completed in type A).

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## 2. CATEGORICAL $\mathfrak{sl}_2$ ACTIONS

**2.1. Notation.** In this paper we work over the complex numbers  $\mathbb{C}$ . Let  $[n] = q^{n-1} + q^{n-3} + \dots + q^{1-n}$  and  $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$  where  $[n]! = [n][n-1] \dots [1]$ .

By a graded category we will mean a category equipped with an auto-equivalence  $\langle 1 \rangle$ . A graded additive  $\mathbb{C}$ -linear 2-category is a category enriched over graded additive  $\mathbb{C}$ -linear categories, that is a 2-category  $\mathcal{K}$  such that the Hom categories  $\text{Hom}_{\mathcal{K}}(A, B)$  between objects  $A$  and  $B$  are graded additive  $\mathbb{C}$ -linear categories (with finite-dimensional Hom spaces) and the composition maps  $\text{Hom}_{\mathcal{K}}(A, B) \times \text{Hom}_{\mathcal{K}}(B, C) \rightarrow \text{Hom}_{\mathcal{K}}(A, C)$  are graded additive  $\mathbb{C}$ -linear functors.

Given a 1-morphism  $A$  in an additive 2-category  $\mathcal{K}$  and a Laurent polynomial  $f = \sum f_a q^a \in \mathbb{N}[q, q^{-1}]$  we write  $\oplus_f A$  for the direct sum over  $a \in \mathbb{Z}$ , of  $f_a$  copies of  $A\langle a \rangle$ . In particular, if  $f = [n]$ , then we write  $\oplus_{[n]} A$  to denote the direct sum  $\oplus_{k=0}^{n-1} A\langle n-1-2k \rangle$ .

An additive category is said to be idempotent complete when every idempotent 1-morphism splits. We say that the additive 2-category  $\mathcal{K}$  is idempotent complete when the Hom categories  $\text{Hom}_{\mathcal{K}}(A, B)$  are idempotent complete for any pair of objects  $A, B$  in  $\mathcal{K}$ , so that all idempotent 2-morphisms split.

If  $\mathcal{A}$  is an abelian category, then we write  $D(\mathcal{A})$  for the bounded derived category of  $\mathcal{A}$ .

**2.2. Categorical actions.** A *categorical  $\mathfrak{sl}_2$  action* with target  $\mathcal{K}$  consists of the following data.

- (i) A graded, additive,  $\mathbb{C}$ -linear, idempotent complete 2-category  $\mathcal{K}$  (the grading shift is denoted  $\langle \cdot \rangle$ ).
- (ii) For each  $\lambda \in \mathbb{Z}$  an object  $\lambda \in \mathcal{K}$ . We write  $\mathcal{K}(\lambda, \lambda') = \text{Hom}_{\mathcal{K}}(\lambda, \lambda')$  for the resulting Hom categories.
- (iii) 1-morphisms  $\mathcal{E}(\lambda) \in \mathcal{K}(\lambda-1, \lambda+1)$  and  $\mathcal{F}(\lambda) \in \mathcal{K}(\lambda+1, \lambda-1)$  for  $\lambda \in \mathbb{Z}$ . For convenience we will sometimes omit the  $\lambda$  when its value is irrelevant.

(iv) 2-morphisms

$$X(\lambda) : \mathcal{E}(\lambda)\langle -1 \rangle \rightarrow \mathcal{E}(\lambda)\langle 1 \rangle \text{ and } T(\lambda) : \mathcal{E}(\lambda+1)\mathcal{E}(\lambda-1)\langle 1 \rangle \rightarrow \mathcal{E}(\lambda+1)\mathcal{E}(\lambda-1)\langle -1 \rangle.$$

On this data we impose the following conditions.

- (i) The objects  $\lambda \in \mathcal{K}$  are zero for  $\lambda \gg 0$  and  $\lambda \ll 0$  (“integrability”).
- (ii) We have isomorphisms  $\mathcal{E}(\lambda)_R \cong \mathcal{F}(\lambda)\langle \lambda \rangle$  and  $\mathcal{E}(\lambda)_L \cong \mathcal{F}(\lambda)\langle -\lambda \rangle$ .
- (iii) If  $\lambda \leq 0$  then

$$\mathcal{F}(\lambda+1)\mathcal{E}(\lambda+1) \cong \mathcal{E}(\lambda-1)\mathcal{F}(\lambda-1) \bigoplus_{[-\lambda]} \text{Id}_\lambda \in \mathcal{K}(\lambda, \lambda)$$

while if  $\lambda \geq 0$  then

$$\mathcal{E}(\lambda-1)\mathcal{F}(\lambda-1) \cong \mathcal{F}(\lambda+1)\mathcal{E}(\lambda+1) \bigoplus_{[\lambda]} \text{Id}_\lambda \in \mathcal{K}(\lambda, \lambda).$$

(iv) The  $X$ s and  $T$ s satisfy the nil affine Hecke relations:

- (a)  $T(\lambda)^2 = 0$
- (b)  $(IT(\lambda-1)) \circ (T(\lambda+1)I) \circ (IT(\lambda-1)) = (T(\lambda+1)I) \circ (IT(\lambda-1)) \circ (T(\lambda+1)I)$  as endomorphisms of  $\mathcal{E}(\lambda+2)\mathcal{E}(\lambda)\mathcal{E}(\lambda-2)$ .
- (c)  $(X(\lambda+1)I) \circ T(\lambda) - T(\lambda) \circ (IX(\lambda-1)) = I = -(IX(\lambda-1)) \circ T(\lambda) + T(\lambda) \circ (X(\lambda+1)I)$  as endomorphisms of  $\mathcal{E}(\lambda+1)\mathcal{E}(\lambda-1)$ .
- (v) For  $\lambda \in \mathcal{K}$  nonzero the endomorphism space  $\text{Hom}(\text{Id}_\lambda, \text{Id}_\lambda\langle i \rangle)$  of the identity 1-morphism  $\text{Id}_\lambda \in \mathcal{K}(\lambda, \lambda)$  is zero if  $i < 0$  and one-dimensional if  $i = 0$ . Moreover, the space of 2-morphisms between any two 1-morphisms is finite dimensional.

**Remark 2.1.** A categorical  $\mathfrak{sl}_2$  action is the same thing as a 2-functor from Rouquier’s 2-category [R] to  $\mathcal{K}$  and by [CL] this is also the same as a 2-functor from Lauda’s 2-category [Ld] to  $\mathcal{K}$ .

**Remark 2.2.** Typically, the 2-category  $\mathcal{K}$  is the 2-category of ( $\mathbb{C}$ -linear, additive, graded) categories where the 1-morphisms are functors and the 2-morphisms are natural transformations. In this paper,  $\mathcal{K}$  will be a 2-category of kernels where the 1-morphisms are kernels and the 2-morphisms are morphisms between kernels.

The action of the affine nilHecke algebra gives us a direct sum decomposition

$$\mathcal{E}^r(\lambda) \cong \bigoplus_{[r]!} \mathcal{E}^{(r)}(\lambda)$$

for some new 1-morphisms  $\mathcal{E}^{(r)}(\lambda) \in \mathcal{K}(\lambda-r, \lambda+r)$  (and similarly we get  $\mathcal{F}^{(r)}(\lambda) \in \mathcal{K}(\lambda+r, \lambda-r)$ ). In most geometric examples (including this paper) these  $\mathcal{E}^{(r)}$  are quite natural and interesting in themselves.

### 3. $\mathcal{D}_{X,h}$ -MODULES AND THE ASSOCIATED GRADED FUNCTOR

**3.1. Notation.** In this section we assume our varieties are smooth, quasi-projective and defined over  $\mathbb{C}$ . For a variety  $X$  we denote by  $\mathcal{O}_X$  the structure sheaf. We write  $\mathcal{O}_X\text{-mod}$  for the category of coherent  $\mathcal{O}_X$ -modules. Similarly, we denote by  $\mathcal{D}_X$  denote the sheaf of differential operators and  $\mathcal{D}_X\text{-mod}$  for the category of coherent  $\mathcal{D}_X$ -modules. All functors will be assumed to be derived unless otherwise noted.

Usually in this paper, we consider  $\mathcal{O}$ -modules on cotangent bundles  $T^*X$ . We write  $\mathcal{O}_{T^*X}\text{-mod}^{\mathbb{C}^\times}$  for the category of  $\mathbb{C}^\times$ -equivariant coherent  $\mathcal{O}_{T^*X}$ -modules on  $T^*X$ , where  $\mathbb{C}^\times$  acts by scaling the fibres with weight 2. We write  $\{1\} : D(\mathcal{O}_{T^*X}\text{-mod}^{\mathbb{C}^\times}) \rightarrow D(\mathcal{O}_{T^*X}\text{-mod}^{\mathbb{C}^\times})$  for the functor given by tensoring by the trivial line bundle carrying a  $\mathbb{C}^\times$ -action of weight 1.

Recall that  $\mathcal{D}_X$  comes equipped with a filtration  $F^0\mathcal{D}_X \subset F^1\mathcal{D}_X \subset \dots$  by the order of differential operator. We let  $\mathcal{D}_{X,h} := \text{Rees}(\mathcal{D}_X) = \bigoplus_k h^k F^k \mathcal{D}_X$  be the associated Rees algebra.  $\mathcal{D}_{X,h}$  is a sheaf of graded algebras which contains functions  $\mathcal{O}_X$  in degree zero and vector fields and the parameter  $h$  in degree 2 subject to the relations  $\xi \cdot f - f \cdot \xi = h\xi(f)$  for all vector fields  $\xi$  and functions  $f$  (we choose to put the vector fields in degree 2 for convenience). We denote by  $\mathcal{D}_{X,h}\text{-mod}$  the category of coherent graded  $\mathcal{D}_{X,h}$ -modules. We write  $\{1\}$  for the functor of grading shift.

If  $(M, F)$  is a  $\mathcal{D}_X$ -module with a compatible filtration then one can form the Rees module  $\text{Rees}(M) = \bigoplus_k h^k F^k M$  which is a  $\mathcal{D}_{X,h}$ -module. It is not hard to check that  $\text{Rees}(M) \otimes_{\mathbb{C}[h]} \mathbb{C}_1 \cong M$  while  $\text{Rees}(M) \otimes_{\mathbb{C}[h]} \mathbb{C}_0 \cong \text{gr}(M)$  is the associated graded of  $M$ .

For studying the associated graded functor from filtered  $\mathcal{D}_X$ -modules to  $\mathcal{O}_{T^*X}$ -modules, it is more convenient to work with the Rees modules. So instead of working with filtered  $\mathcal{D}_X$ -modules we will generally work with  $\mathcal{D}_{X,h}$ -modules, rather than filtered  $\mathcal{D}$ -modules.

By the definition of  $\mathcal{D}_{X,h}$  we have that

$$\mathcal{D}_{X,h} \otimes_{\mathbb{C}[h]} \mathbb{C}_0 \cong \pi_* \mathcal{O}_{T^*X}$$

(where  $\pi : T^*X \rightarrow X$  is the natural projection and  $\mathbb{C}_0 = \mathbb{C}[h]/h$ ). Therefore we have a functor

$$D(\mathcal{D}_{X,h}\text{-mod}) \rightarrow D(\pi_* \mathcal{O}_{T^*X}\text{-mod}), \quad M \rightarrow M \otimes_{\mathbb{C}[h]} \mathbb{C}_0.$$

Since localization gives an equivalence  $\pi_* \mathcal{O}_{T^*X}\text{-mod} \rightarrow \mathcal{O}_{T^*X}\text{-mod}$  we get a functor

$$\text{gr} : D(\mathcal{D}_{X,h}\text{-mod}) \rightarrow D(\mathcal{O}_{T^*X}\text{-mod}^{\mathbb{C}^\times}).$$

Note that if  $M \in \mathcal{D}_{X,h}\text{-mod}$ , then  $M$  is graded by definition and so  $M \otimes_{\mathbb{C}[h]} \mathbb{C}_0$  will be a graded  $\pi_* \mathcal{O}_{T^*X}$ -module, which is the same thing as a  $\mathbb{C}^\times$ -equivariant  $\mathcal{O}_{T^*X}$ -module. Finally, note that  $\text{gr}(M\{1\}) = \text{gr}(M)\{1\}$ .

Note that one also has a similar but simpler functor  $\mathcal{D}_{X,h}\text{-mod} \rightarrow \mathcal{D}_X\text{-mod}$  given by  $M \rightarrow M \otimes_{\mathbb{C}[h]} \mathbb{C}_1$ . (In this case, setting  $h$  to be 1 turns the grading on  $M$  into a filtration.)

**Remark 3.1.** There are three grading shifts that show up in this paper. The first is the internal shift  $\{1\}$  mentioned above. The second is the cohomological grading shift  $[1]$ . The third is  $\langle 1 \rangle := [1]\{-1\}$  which appears naturally when working with the categorical  $\mathfrak{sl}_2$  action (as well as in other contexts such as IC Hodge modules).

In this section we will explain how the functor  $\text{gr}$  commutes with pullbacks and pushforwards of  $\mathcal{D}_h$ -modules. These results first appeared in a paper by Laumon [Lm] using the language of exact categories and filtered modules. We reinterpret and reprove these results using the language of  $\mathcal{D}_h$ -modules.

**3.2. Pull-back.** We begin with a discussion of the pullback functor in the setting of  $\mathcal{D}$ -modules. We consider a map  $f : X \rightarrow Y$  between smooth varieties. For a  $\mathcal{D}_Y$ -module  $N$  we have the usual coherent pullback  $f^*N$ , which is then a  $f^*\mathcal{D}_Y$ -module. Since  $f$  induces

a natural morphism  $\mathcal{T}_X \rightarrow f^*\mathcal{T}_Y$  the sheaf  $f^*N$  acquires an action of  $\mathcal{D}_X$ . Following [HTT, Sect. 1.3] we rewrite this functor as

$$f^*N \cong f^*\mathcal{D}_Y \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}N.$$

where the  $\mathcal{D}_X$  action on the tensor product is induced from that on  $f^*\mathcal{D}_Y$ . Then the extension to  $\mathcal{D}_h$ -modules is clear we define

$$f^*N := f^*\mathcal{D}_{Y,h} \otimes_{f^{-1}\mathcal{D}_{Y,h}} f^{-1}N.$$

where  $f^*\mathcal{D}_{Y,h}$  is a  $\mathcal{D}_{X,h}$  module again by the map  $\mathcal{T}_X \rightarrow f^*\mathcal{T}_Y$ . This functor is left exact and cohomologically bounded. Therefore, we can define  $f^\dagger : D(\mathcal{D}_{Y,h}\text{-mod}) \rightarrow D(\mathcal{D}_{X,h}\text{-mod})$  by

$$f^\dagger N := f^*N[\dim(X) - \dim(Y)].$$

Recall that  $f^\dagger$  is useful since it appears in the base change formula for  $\mathcal{D}$ -modules.

To describe what happens when we specialize to  $h = 0$  consider the following commutative diagram associated to  $f : X \rightarrow Y$ .

$$(1) \quad \begin{array}{ccccc} T^*X & \xleftarrow{\pi_f} & X \times_Y T^*Y & \xrightarrow{p_1} & X \\ \downarrow \pi_X & \nearrow p_1 & \downarrow p_2 & & \downarrow f \\ X & & T^*Y & \xrightarrow{\pi_Y} & Y \end{array}$$

**Proposition 3.2.** *For  $N \in D(\mathcal{D}_{Y,h}\text{-mod})$  we have  $\mathrm{gr}(f^*N) \cong \pi_{f*}p_2^*(\mathrm{gr}N)$ .*

*Proof.* First, we have

$$\begin{aligned} \pi_{X*}(f^*N \otimes_{\mathbb{C}[h]} \mathbb{C}_0) &\cong \pi_{X*}[(f^*\mathcal{D}_{Y,h} \otimes_{\mathbb{C}[h]} \mathbb{C}_0) \otimes_{f^{-1}\mathcal{D}_{Y,h} \otimes_{\mathbb{C}[h]} \mathbb{C}_0} (f^{-1}(N) \otimes_{\mathbb{C}[h]} \mathbb{C}_0)] \\ &\cong f^*\mathrm{Sym}^*T_Y \otimes_{f^{-1}\mathrm{Sym}^*T_Y} f^{-1}\pi_{Y*}(N \otimes_{\mathbb{C}[h]} \mathbb{C}_0) \\ &\cong \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathrm{Sym}^*T_Y \otimes_{f^{-1}\mathrm{Sym}^*T_Y} f^{-1}\pi_{Y*}(N \otimes_{\mathbb{C}[h]} \mathbb{C}_0) \\ &\cong \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\pi_{Y*}(N \otimes_{\mathbb{C}[h]} \mathbb{C}_0) \\ &\cong f^*\pi_{Y*}(N \otimes_{\mathbb{C}[h]} \mathbb{C}_0) \end{aligned}$$

where the second line is a  $\mathrm{Sym}^*T_X$ -module via the natural map  $\mathrm{Sym}^*T_X \rightarrow f^*\mathrm{Sym}^*T_Y$  and, also in the second line and beyond, we think of  $f^{-1}\pi_{Y*}(N \otimes_{\mathbb{C}[h]} \mathbb{C}_0)$  as a  $f^{-1}\mathrm{Sym}^*T_Y$ -module. Next, since  $\pi_Y$  is flat, the commutativity of the square in (1) implies

$$f^*\pi_{Y*}(N \otimes_{\mathbb{C}[h]} \mathbb{C}_0) \cong p_{1*}p_2^*(N \otimes_{\mathbb{C}[h]} \mathbb{C}_0).$$

The result follows since  $p_{1*} = \pi_{X*} \circ \pi_{f*}$ .  $\square$

**3.3. Push-forward.** Consider a morphism  $f : X \rightarrow Y$ . Let us recall the usual push-forward of a  $\mathcal{D}_X$ -module  $M$

$$(2) \quad \int_f M := Rf_*(\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^L M) \in D(\mathcal{D}_Y\text{-mod})$$

where  $Rf_*$  is the usual derived pushforward of quasi-coherent  $\mathcal{O}$ -modules and  $\mathcal{D}_{Y \leftarrow X}$  is the transfer bimodule, defined as  $f^*(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \omega_Y^{-1}) \otimes_{\mathcal{O}_X} \omega_X$  (where  $f^*$  is the usual pullback).

Note that  $\mathcal{D}_{Y \leftarrow X}$  carries the structure of a right  $\mathcal{D}_X$ -module as follows.  $\mathcal{D}_Y$  is naturally a right  $\mathcal{D}_Y$ -module by multiplication and so  $\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \omega_Y^{-1}$  is a left  $\mathcal{D}_Y$ -module by side-changing.



Then  $f^*(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \omega_Y^{-1})$  is naturally a left  $\mathcal{D}_X$ -module by pull-back. Thus by side-changing  $f^*(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \omega_Y^{-1}) \otimes_{\mathcal{O}_X} \omega_X$  is naturally a right  $\mathcal{D}_X$ -module. Thus, the tensor product in (2) makes sense. Moreover, the left  $\mathcal{D}_Y$ -module structure in (2) is inherited from the left  $\mathcal{D}_Y$ -module structure on  $\mathcal{D}_Y$ .

Let us consider the  $\mathcal{D}_h$  analogue of this construction. The issue is to define the analogue of  $\mathcal{D}_{Y \leftarrow X}$ . For this we simply note that  $f^*(\mathcal{D}_{Y,h} \otimes_{\mathcal{O}_Y} \omega_Y^{-1}) \otimes_{\mathcal{O}_X} \omega_X$  is naturally a right  $\mathcal{D}_{X,h}$ -module by the same consideration as above. We denote this module by  $\mathcal{D}_{Y \leftarrow X,h}$ . Then we define

$$(3) \quad \int_f M := Rf_*(\mathcal{D}_{Y \leftarrow X,h} \otimes_{\mathcal{D}_{X,h}}^L M) \in D(\mathcal{D}_{Y,h}\text{-mod})$$

which is naturally a left  $\mathcal{D}_{Y,h}$ -module via the left action of  $\mathcal{D}_{Y,h}$  on itself.

Now we look at what happens when we specialize to  $h = 0$ . As before we use the diagram

$$(4) \quad \begin{array}{ccc} X \times_Y T^*Y & \xrightarrow{p_1} & X \\ \downarrow & & \downarrow f \\ T^*Y & \xrightarrow{\pi_Y} & Y \end{array}$$

**Proposition 3.3.** *As  $\mathcal{O}_X$ -modules we have*

$$\mathcal{D}_{Y \leftarrow X,h} \otimes_{\mathbb{C}[h]} \mathbb{C}_0 \cong Rp_{1*}(\mathcal{O}_{X \times_Y T^*Y}) \otimes_{\mathcal{O}_X} \omega_{X/Y}$$

where  $\omega_{X/Y}$  denotes  $\omega_X \otimes_{\mathcal{O}_X} f^* \omega_Y^{-1}$ .

*Proof.* We recall that  $\mathcal{D}_{Y \leftarrow X,h} = f^*(\mathcal{D}_{Y,h} \otimes_{\mathcal{O}_Y} \omega_Y^{-1}) \otimes_{\mathcal{O}_X} \omega_X$ . Now, by definition we have  $\mathcal{D}_{Y,h} \otimes_{\mathbb{C}[h]} \mathbb{C}_0 \cong \pi_{Y*} \mathcal{O}_{T^*Y}$ . So it follows that

$$\mathcal{D}_{Y \leftarrow X,h} \otimes_{\mathbb{C}[h]} \mathbb{C}_0 \cong f^*(\pi_{Y*}(\mathcal{O}_{T^*Y})) \otimes \omega_{X/Y}$$

Therefore, the result will follow if the base change map  $f^*(\pi_{Y*}(\mathcal{O}_{T^*Y})) \rightarrow Rp_*(\mathcal{O}_{X \times_Y T^*Y})$  is an isomorphism. To prove this, we note that the map  $\pi_Y$  is affine, and therefore so is  $p_1$ . Thus the question is reduced to the case of a morphism of affine schemes  $\text{Spec}(B) \rightarrow \text{Spec}(A)$ . This gives a diagram of ring morphisms

$$(5) \quad \begin{array}{ccc} B \otimes_A \text{Sym}_A(\mathcal{T}_A) & \longleftarrow & B \\ \uparrow & & \uparrow \\ \text{Sym}_A(\mathcal{T}_A) & \longleftarrow & A \end{array}$$

and now the result follows immediately from the fact the  $\text{Sym}_A(\mathcal{T}_A)$  is flat over  $A$ .  $\square$

**Corollary 3.4.** *For  $M \in D(\mathcal{D}_{X,h}\text{-mod})$  and using the notation from (1) we have*

$$\text{gr} \left( \int_f M \right) \cong p_{2*}(\pi_f^* \text{gr}(M) \otimes p_1^* \omega_{X/Y}).$$



*Proof.* Using Proposition 3.3, we have

$$\begin{aligned} \left( \int_f M \right) \otimes_{\mathbb{C}[h]} \mathbb{C}_0 &\cong f_*(\mathcal{D}_{Y \leftarrow X, h} \otimes_{\mathcal{D}_{X, h}} M) \otimes_{\mathbb{C}[h]} \mathbb{C}_0 \\ &\cong f_*[p_{1*}(\mathcal{O}_{X \times_Y T^*Y}) \otimes_{\mathcal{O}_X} \omega_{X/Y} \otimes_{\mathcal{O}_X} (M \otimes_{\mathbb{C}[h]} \mathbb{C}_0)] \\ &\cong f_*p_{1*}[p_1^* \omega_{X/Y} \otimes_{\mathcal{O}_{X \times_Y T^*Y}} p_1^*(M \otimes_{\mathbb{C}[h]} \mathbb{C}_0)] \end{aligned}$$

where in the second and third lines we view  $M \otimes_{\mathbb{C}[h]} \mathbb{C}_0$  as a  $\mathrm{Sym}^* T_X$ -module on  $X$ . Thus we get

$$\left( \int_f M \right) \otimes_{\mathbb{C}[h]} \mathbb{C}_0 \cong \pi_{Y*} p_{2*}(\pi_f^*(M \otimes_{\mathbb{C}[h]} \mathbb{C}_0) \otimes_{\mathcal{O}_{X \times_Y T^*Y}} p_1^* \omega_{X/Y}).$$

The result follows.  $\square$

If  $f : X \rightarrow Y$  is an inclusion, then we will use the notation  $\delta_{X, h} := \int_f \mathcal{O}_{X, h} \{\dim X\}$ . This is sometimes called the sheaf of “delta-functions” along  $X$ .

**Corollary 3.5.** *Assume that  $f : X \rightarrow Y$  is an inclusion. Then*

$$\mathrm{gr}(\delta_{X, h}) \cong \mathcal{O}_{T_X^*(Y)} \otimes p^* \omega_{X/Y} \{\dim X\}$$

where  $T_X^*(Y) \subset (T^*Y)|_X$  denotes the conormal bundle of  $X \subset Y$  and  $p : T_X^*(Y) \rightarrow X$  is the natural projection.

*Proof.* This follows immediately from Corollary 3.4 by observing that  $\pi_f^{-1}(X) = T_X^*(Y)$ .  $\square$

**3.4. Tensor products.** Fix a variety  $X$  as before and let  $M, N \in D(\mathcal{D}_{X, h})$ . Then one can consider the (derived) tensor product  $M \otimes_{\mathcal{O}_{X, h}} N \in D(\mathcal{D}_{X, h})$  where  $\mathcal{D}_{X, h}$  acts on the tensor using the Leibniz rule. Alternatively, we have  $M \otimes_{\mathcal{O}_{X, h}} N = \Delta^*(M \boxtimes N)$  where  $\Delta : X \rightarrow X \times X$  is the diagonal inclusion and  $M \boxtimes N$  is the exterior tensor product. We now explain how this tensor product commutes with the associated graded functor.

Denote by  $\rho : T^*X \times_X T^*X \rightarrow T^*X$  the map which takes  $(p, v_1, v_2) \in T^*X \times_X T^*X$  to  $(p, v_1 + v_2)$  where  $v_1$  and  $v_2$  are covectors over  $p \in X$ . We also have the two natural projections  $\pi_1, \pi_2 : T^*X \times_X T^*X \rightarrow T^*X$ .

**Proposition 3.6.** *Let  $M, N \in D(\mathcal{D}_{X, h})$ . Then*

$$\mathrm{gr}(M \otimes_{\mathcal{O}_{X, h}} N) \cong \rho_*(\pi_1^*(\mathrm{gr} M) \otimes \pi_2^*(\mathrm{gr} N)).$$

*Proof.* By Proposition 3.2 we have

$$(\Delta^*(M \boxtimes N)) \otimes_{\mathbb{C}[h]} \mathbb{C}_0 \cong \rho_* p_2^*((M \boxtimes N) \otimes_{\mathbb{C}[h]} \mathbb{C}_0)$$

where we use the natural maps

$$T^*X \xleftarrow{\rho} X_{X \times X} T^*(X \times X) \xrightarrow{p_2} T^*X \times T^*X.$$

To see why the map  $\pi_\Delta$  in (1) corresponds to  $\rho$  note that  $\pi_\Delta$  is induced by  $TX \rightarrow T(X \times X)|_\Delta$  via the diagonal map and it is easy to see that the dual of this map is  $\rho$ .

Thus we get

$$\begin{aligned}
(M \otimes_{\mathcal{O}_{X,h}} N) \otimes_{\mathbb{C}[h]} \mathbb{C}_0 &\cong (\Delta^*(M \boxtimes N)) \otimes_{\mathbb{C}[h]} \mathbb{C}_0 \\
&\cong \rho_* p_2^*((M \boxtimes N) \otimes_{\mathbb{C}[h]} \mathbb{C}_0) \\
&\cong \rho_* p_2^*((M \otimes_{\mathbb{C}[h]} \mathbb{C}_0) \boxtimes (N \otimes_{\mathbb{C}[h]} \mathbb{C}_0)) \\
&\cong \rho_*(\pi_1^*(M \otimes_{\mathbb{C}[h]} \mathbb{C}_0) \otimes \pi_2^*(N \otimes_{\mathbb{C}[h]} \mathbb{C}_0))
\end{aligned}$$

where the second last isomorphism follows from  $(M \boxtimes N) \otimes_{\mathbb{C}[h]} \mathbb{C}_0 \cong (M \otimes_{\mathbb{C}[h]} \mathbb{C}_0) \boxtimes (N \otimes_{\mathbb{C}[h]} \mathbb{C}_0)$ .  $\square$

We will also find it useful to define the tensor product with respect to  $\dagger$ . Namely we define  $M \otimes_{\mathcal{O}_{X,h}}^\dagger N$  as  $\Delta^\dagger(M \boxtimes N)$ . Note that  $M \otimes_{\mathcal{O}_{X,h}}^\dagger N = M \otimes_{\mathcal{O}_{X,h}} N[-\dim(X)]$ .

**Corollary 3.7.** *Suppose that  $Y$  and  $Z$  are two smooth subvarieties of  $X$  which intersect transversally. Then we have  $\delta_{Y,h} \otimes_{\mathcal{O}_{X,h}}^\dagger \delta_{Z,h} \cong \delta_{Y \cap Z,h} \langle -\dim(X) \rangle$ .*

*Proof.* The main thing to show is that  $\mathcal{O}_{Y,h} \otimes_{\mathcal{O}_{X,h}} \mathcal{O}_{Z,h}$  is supported in cohomological degree zero (i.e. there are no derived terms). When  $h \neq 0$  this tensor product restricts to the usual tensor product of  $\mathcal{D}$ -modules, which is exact. So it remains to show that

$$(\mathcal{O}_{Y,h} \otimes_{\mathcal{O}_{X,h}} \mathcal{O}_{Z,h}) \otimes_{\mathbb{C}[h]} \mathbb{C}_0 = \text{gr}(\mathcal{O}_{Y,h} \otimes_{\mathcal{O}_{X,h}} \mathcal{O}_{Z,h})$$

is supported in degree zero.

By Proposition 3.6 it suffices to show that  $\pi_1^* \text{gr}(\mathcal{O}_{Y,h}) \otimes \pi_2^* \text{gr}(\mathcal{O}_{Z,h})$  is supported in degree zero (since  $\rho$  is affine) where  $\pi_1, \pi_2$  are the two projections from  $T^*X \times_X T^*X$  to  $T^*X$ . On the other hand (ignoring shifts),  $\text{gr}(\mathcal{O}_{Y,h}) \cong \mathcal{O}_{T_Y^*(X)} \otimes p^* \omega_{Y/X}$  and likewise for  $\text{gr}(\mathcal{O}_{Z,h})$ . Thus, up to tensoring by a global line bundle,  $\pi_1^* \text{gr}(\mathcal{O}_{Y,h}) \otimes \pi_2^* \text{gr}(\mathcal{O}_{Z,h})$  is equal to

$$\pi_1^* \mathcal{O}_{T_Y^*(X)} \otimes \pi_2^* \mathcal{O}_{T_Z^*(X)} \cong \mathcal{O}_{\pi_1^{-1}(T_Y^*(X))} \otimes \mathcal{O}_{\pi_2^{-1}(T_Z^*(X))} \cong \mathcal{O}_{T_{Y \cap Z}^*(X)} \in D(\mathcal{O}_{T^*X \times_X T^*X})$$

where, to get the second isomorphism, we are using the condition that  $Y$  and  $Z$  intersect transversally.

Finally, tracing through the grading  $[\cdot]$  and  $\{\cdot\}$  shifts gives the desired result.  $\square$

**Corollary 3.8.** *Suppose  $f : X \rightarrow Y$  is a morphism between smooth varieties and  $Z \subset Y$  is smooth. If  $f^{-1}(Z) \subset X$  is smooth of the expected dimension then*

$$f^\dagger(\delta_{Z,h}) \cong \delta_{f^{-1}(Z),h} \langle \dim(X) - \dim(Y) \rangle.$$

*Proof.* The expected dimension hypothesis implies that  $f^* \mathcal{O}_{Z,h} \cong \mathcal{O}_{f^{-1}(Z),h}$  (apply Corollary 3.7 to the graph of  $f$  and  $X \times f^{-1}(Z)$  inside  $X \times Y$ ). So we have

$$\begin{aligned}
f^\dagger(\delta_{Z,h}) &\cong f^* \mathcal{O}_{Z,h} [\dim(X) - \dim(Y)] \{\dim(Z)\} \\
&\cong \mathcal{O}_{f^{-1}(Z),h} \{\dim(f^{-1}(Z))\} [\dim(X) - \dim(Y)] \{-\dim(X) + \dim(Y)\} \\
&\cong \delta_{f^{-1}(Z),h} \langle \dim(X) - \dim(Y) \rangle
\end{aligned}$$

where we used that  $\dim(f^{-1}(Z)) = \dim(X) - \dim(Y) + \dim(Z)$ .  $\square$

#### 4. KERNELS AND THE ASSOCIATED GRADED FUNCTOR

In this section we recall and develop the theory of kernels for  $\mathcal{O}_X$  and  $\mathcal{D}_{X,h}$  modules.

**4.1. Kernels for  $\mathcal{O}_X$ -modules.** First recall the formalism of kernels for  $\mathcal{O}_X$ -modules. Let  $X, Y$  be two smooth varieties. A kernel is any object  $\mathcal{P} \in D(\mathcal{O}_{X \times Y}\text{-mod})$ . It defines the associated integral (or Fourier-Mukai) transform

$$\begin{aligned} \Phi_{\mathcal{P}} : D(\mathcal{O}_X\text{-mod}) &\rightarrow D(\mathcal{O}_Y\text{-mod}) \\ M &\mapsto \pi_{2*}(\pi_1^*(M) \otimes \mathcal{P}) \end{aligned}$$

where  $\pi_1$  and  $\pi_2$  are the projections from  $X \times Y$  to  $X$  and  $Y$  respectively. The left and right adjoints of  $\Phi_{\mathcal{P}}$  are isomorphic to the functors induced by the kernels

$$\mathcal{P}_L := \mathcal{P}^\vee \otimes \pi_1^* \omega_Y[\dim(Y)] \text{ and } \mathcal{P}_R := \mathcal{P}^\vee \otimes \pi_2^* \omega_X[\dim(X)]$$

respectively, where  $\mathcal{P}^\vee \in D(\mathcal{O}_{Y \times X}\text{-mod})$  denotes the derived dual of  $\mathcal{P}$ .

We can also express composition of functors in terms of their kernels. If  $X, Y, Z$  are varieties and  $\Phi_{\mathcal{P}} : D(\mathcal{O}_X\text{-mod}) \rightarrow D(\mathcal{O}_Y\text{-mod})$ ,  $\Phi_{\mathcal{Q}} : D(\mathcal{O}_Y\text{-mod}) \rightarrow D(\mathcal{O}_Z\text{-mod})$  then  $\Phi_{\mathcal{Q}} \circ \Phi_{\mathcal{P}}$  is induced by the kernel

$$\mathcal{Q} * \mathcal{P} := \pi_{13*}(\pi_{12}^*(\mathcal{P}) \otimes \pi_{23}^*(\mathcal{Q}))$$

where  $*$  is called the convolution product. We now describe the analogous formalism for  $\mathcal{D}_h$ -modules.

**4.2. Kernels for  $\mathcal{D}_{X,h}$ -modules.** Consider again two smooth varieties  $X$  and  $Y$  and an object  $\mathcal{P} \in D(\mathcal{D}_{X \times Y, h}\text{-mod})$ . Such a kernel induces a functor  $\Phi_{\mathcal{P}} : D(\mathcal{D}_{X, h}\text{-mod}) \rightarrow D(\mathcal{D}_{Y, h}\text{-mod})$  via

$$\Phi_{\mathcal{P}}(M) = \int_{p_2} p_1^\dagger M \otimes_{\mathcal{O}_{X \times Y, h}}^\dagger \mathcal{P}$$

where  $M \in D(\mathcal{D}_{X, h}\text{-mod})$ . As above, if  $\mathcal{P} \in D(\mathcal{D}_{X \times Y, h}\text{-mod})$  and  $\mathcal{Q} \in D(\mathcal{D}_{Y \times Z, h}\text{-mod})$  then we can define

$$\mathcal{Q} * \mathcal{P} := \int_{p_{13}} p_{12}^\dagger(\mathcal{P}) \otimes_{\mathcal{O}_{X \times Y \times Z, h}}^\dagger p_{23}^\dagger(\mathcal{Q}).$$

The following result follows from a formal argument which is exactly the same as in the case of quasi-coherent  $\mathcal{O}$ -modules.

**Lemma 4.1.** *If  $\mathcal{P} \in D(\mathcal{D}_{X \times Y, h}\text{-mod})$  and  $\mathcal{Q} \in D(\mathcal{D}_{Y \times Z, h}\text{-mod})$  then  $\Phi_{\mathcal{Q}} \circ \Phi_{\mathcal{P}} \cong \Phi_{\mathcal{Q} * \mathcal{P}}$ .*

**4.3. The associated graded of kernels.** For a kernel  $\mathcal{P} \in D(\mathcal{D}_{X \times Y, h}\text{-mod})$  we define the “directed” associated graded by

$$(6) \quad \vec{\text{gr}}(\mathcal{P}) := (1 \times \iota)^* \text{gr}(\mathcal{P}) \otimes \omega_Y[-\dim(X)] \in D(\mathcal{O}_{T^*X \times T^*Y}\text{-mod})$$

where  $\iota : T^*Y \rightarrow T^*Y$  is the involution acting by multiplication by  $(-1)$  on the fibres and  $\omega_Y$  is the canonical bundle pulled back from  $Y$  via the natural projection  $T^*X \times T^*Y \rightarrow Y$ . We say “directed” because  $\vec{\text{gr}}(\mathcal{P})$  depends on whether we think of  $\mathcal{P}$  as an object on  $X \times Y$  or  $Y \times X$ .

**Proposition 4.2.** *Let  $X, Y, Z$  be smooth varieties with  $\mathcal{P} \in D(\mathcal{D}_{X \times Y, h}\text{-mod})$  and  $\mathcal{Q} \in D(\mathcal{D}_{Y \times Z, h}\text{-mod})$ . Then  $\vec{\text{gr}}(\mathcal{Q} * \mathcal{P}) \cong \vec{\text{gr}}(\mathcal{Q}) * \vec{\text{gr}}(\mathcal{P})$ .*

*Proof.* We will write  $d_X, d_Y, d_Z$  for the dimensions of  $X, Y, Z$ . We begin by first calculating  $\text{gr}(p_{12}^\dagger(\mathcal{P}) \otimes_{\mathcal{O}_{X \times Y \times Z, h}} p_{23}^\dagger(\mathcal{Q}))$ . By Proposition 3.6 we have that

$$\text{gr}(p_{12}^\dagger(\mathcal{P}) \otimes_{X \times Y \times Z, h} p_{23}^\dagger(\mathcal{Q})) \cong \rho_* \pi_1^*(\text{gr}(p_{12}^\dagger \mathcal{P})) \otimes \pi_2^*(\text{gr}(p_{23}^\dagger \mathcal{Q}))$$

where

$$\pi_1, \pi_2, \rho : T^*(X \times Y \times Z) \times_{X \times Y \times Z} T^*(X \times Y \times Z) \rightarrow T^*(X \times Y \times Z)$$

are the natural maps as in section 3.4.

Now by Proposition 3.2 we have  $\text{gr}(p_{12}^\dagger \mathcal{P}) \cong i_{12*} \pi_{12}^*(\text{gr} \mathcal{P})[d_Z]$  where  $i_{12}$  and  $\pi_{12}$  are the natural inclusion and projection maps  $T^*(Y \times X \times Z) \xleftarrow{i_{12}} T^*(X \times Y) \times Z \xrightarrow{\pi_{12}} T^*(X \times Y)$ . Similarly  $\text{gr}(p_{23}^\dagger \mathcal{Q}) \cong i_{23*} \tilde{\pi}_{23}^*(\text{gr} \mathcal{Q})[d_X]$  so we get

$$\text{gr}(p_{12}^\dagger(\mathcal{P}) \otimes_{\mathcal{O}_{X \times Y \times Z, h}} p_{23}^\dagger(\mathcal{Q})) \cong \rho_*(\pi_1^* i_{12*} \tilde{\pi}_{12}^*(\text{gr} \mathcal{P}) \otimes \pi_2^* i_{23*} \tilde{\pi}_{23}^*(\text{gr} \mathcal{Q}))[-d_Y].$$

Next we take a fibre product to get

$$\begin{array}{ccc} [T^*(X \times Y) \times Y] \times_{X \times Y \times Z} T^*(X \times Y \times Z) & \xrightarrow{\tilde{i}_{12}} & T^*(X \times Y \times Z) \times_{X \times Y \times Z} T^*(X \times Y \times Z) \\ \downarrow \tilde{\pi}_1 & & \downarrow \pi_1 \\ T^*(X \times Y) \times Z & \xrightarrow{i_{12}} & T^*(X \times Y) \times T^*Z \end{array}$$

Since  $\pi_1$  is flat we get that  $\pi_1^* i_{12*}(\cdot) \cong \tilde{i}_{12*} \tilde{\pi}_1^*(\cdot)$ . Similarly we get  $\pi_2^* i_{23*}(\cdot) \cong \tilde{i}_{23*} \tilde{\pi}_2^*(\cdot)$ . Putting this together gives

$$\begin{aligned} \text{gr}(p_{12}^\dagger(\mathcal{P}) \otimes_{\mathcal{O}_{X \times Y \times Z, h}} p_{23}^\dagger(\mathcal{Q})) &\cong \rho_*(\tilde{i}_{12*} \tilde{\pi}_1^* \tilde{\pi}_{12}^*(\text{gr} \mathcal{P}) \otimes \tilde{i}_{23*} \tilde{\pi}_2^* \tilde{\pi}_{23}^*(\text{gr} \mathcal{Q}))[-d_Y] \\ &\cong \rho_*(\tilde{i}_{12*}(\tilde{\pi}_1^* \tilde{\pi}_{12}^*(\text{gr} \mathcal{P}) \otimes \tilde{i}_{12}^* \tilde{i}_{23*} \tilde{\pi}_2^* \tilde{\pi}_{23}^*(\text{gr} \mathcal{Q})))[-d_Y]. \end{aligned}$$

To compute  $\tilde{i}_{12}^* \tilde{i}_{23*}(\cdot)$  we use the following fibre product

$$\begin{array}{ccc} [T^*(X \times Y) \times Z] \times_{X \times Y \times Z} [X \times T^*(Y \times Z)] & \xrightarrow{\hat{i}_{12}} & T^*(X \times Y \times Z) \times_{X \times Y \times Z} [X \times T^*(Y \times Z)] \\ \downarrow \hat{i}_{23} & & \downarrow \tilde{i}_{23} \\ [T^*(X \times Y) \times Z] \times_{X \times Y \times Z} T^*(X \times Y \times Z) & \xrightarrow{\tilde{i}_{12}} & T^*(X \times Y \times Z) \times_{X \times Y \times Z} T^*(X \times Y \times Z). \end{array}$$

Since the images of  $\hat{i}_{12}$  and  $\tilde{i}_{23}$  meet transversely we have  $\tilde{i}_{12}^* \tilde{i}_{23*}(\cdot) \cong \hat{i}_{23*} \hat{i}_{12}^*(\cdot)$  and so we get

$$\begin{aligned} \text{gr}(p_{12}^\dagger(\mathcal{P}) \otimes_{\mathcal{O}_{X \times Y \times Z, h}} p_{23}^\dagger(\mathcal{Q})) &\cong \rho_* \tilde{i}_{12*}(\tilde{\pi}_1^* \tilde{\pi}_{12}^*(\text{gr} \mathcal{P}) \otimes \hat{i}_{23*} \hat{i}_{12}^* \tilde{\pi}_2^* \tilde{\pi}_{23}^*(\text{gr} \mathcal{Q}))[-d_Y] \\ &\cong \rho_* \tilde{i}_{12*} \hat{i}_{23*}(\hat{i}_{23}^* \tilde{\pi}_1^* \tilde{\pi}_{12}^*(\text{gr} \mathcal{P}) \otimes \hat{i}_{12}^* \tilde{\pi}_2^* \tilde{\pi}_{23}^*(\text{gr} \mathcal{Q}))[-d_Y]. \end{aligned}$$

Now  $\tilde{\pi}_{12} \circ \tilde{p}_1 \circ \hat{i}_{12}$  is precisely the projection

$$q_{12} : [T^*(X \times Y) \times Z] \times_{X \times Y \times Z} [X \times T^*(Y \times Z)] \rightarrow T^*(X \times Y)$$

onto the first factor while  $\tilde{\pi}_{23} \circ \tilde{p}_2 \circ \hat{i}_{23}$  is the projection

$$q_{23} : [T^*(X \times Y) \times Z] \times_{X \times Y \times Z} [X \times T^*(Y \times Z)] \rightarrow T^*(Y \times Z).$$

Also

$$q_{13} := \rho \circ \tilde{i}_{12} \circ \hat{i}_{23} : T^*(X \times Z) \times [T^*Y \times_Y T^*Y] \rightarrow T^*(X \times Z) \times T^*Y$$

is just the map  $1 \times \rho_Y$  where  $\rho_Y : T^*Y \times_Y T^*Y \rightarrow T^*Y$  is the map from section 3.4. So we get

$$\mathrm{gr}(p_{12}^\dagger(\mathcal{P}) \otimes_{\mathcal{O}_{X \times Y \times Z, h}}^\dagger p_{23}^\dagger(\mathcal{Q})) \cong q_{13*}(q_{12}^*(\mathrm{gr}\mathcal{P}) \otimes q_{23}^*(\mathrm{gr}\mathcal{Q}))[-d_Y].$$

Finally, we have

$$\begin{aligned} \vec{\mathrm{gr}}(\mathcal{Q} * \mathcal{P}) &\cong (1 \times \iota)^* \mathrm{gr}(p_{13*}(p_{12}^\dagger(\mathcal{P}) \otimes_{\mathcal{O}_{X \times Y \times Z, h}}^\dagger p_{23}^\dagger(\mathcal{Q}))) \otimes \omega_Z[-d_X] \\ &\cong (1 \times \iota)^* \tilde{\pi}_{13*}(i_{13}^*(\mathrm{gr}(p_{12}^\dagger(\mathcal{P}) \otimes_{\mathcal{O}_{X \times Y \times Z, h}}^\dagger p_{23}^\dagger(\mathcal{Q}))) \otimes \omega_Y) \otimes \omega_Z[-d_X] \end{aligned}$$

where  $i_{13}$  and  $\tilde{\pi}_{13}$  are the natural maps  $T^*(X \times Y \times Z) \xleftarrow{i_{13}} T^*(X \times Z) \times Y \xrightarrow{\tilde{\pi}_{13}} T^*(X \times Z)$ . Using the above result this gives

$$\vec{\mathrm{gr}}(\mathcal{Q} * \mathcal{P}) \cong (1 \times \iota)^* \tilde{\pi}_{13*}(i_{13}^* q_{13*}(q_{12}^*(\mathrm{gr}\mathcal{P}) \otimes q_{23}^*(\mathrm{gr}\mathcal{Q})) \otimes \omega_Y) \otimes \omega_Z[-d_X - d_Y].$$

To compute  $i_{13}^* q_{13*}(\cdot)$  we use the following fibre diagram

$$\begin{array}{ccc} T^*(X \times Z) \times T^*Y & \xrightarrow{\tilde{i}_{13}} & T^*(X \times Z) \times [T^*Y \times_Y T^*Y] \\ \downarrow \tilde{q}_{13} & & \downarrow q_{13} \\ T^*(X \times Z) \times Y & \xrightarrow{i_{13}} & T^*(X \times Z) \times T^*Y \end{array}$$

where  $\tilde{i}_{13}$  is induced by the map  $T^*Y \rightarrow T^*Y \times_Y T^*Y$  which takes  $(p, v) \mapsto (p, -v, v)$ . Since  $q_{13}$  is flat we get  $i_{13}^* q_{13*}(\cdot) \cong \tilde{q}_{13*} \tilde{i}_{13}^*(\cdot)$  and hence

$$\vec{\mathrm{gr}}(\mathcal{Q} * \mathcal{P}) \cong (1 \times \iota)^* \tilde{\pi}_{13*}(\tilde{q}_{13*}(\tilde{i}_{13}^* q_{12}^*(\mathrm{gr}\mathcal{P}) \otimes \tilde{i}_{13}^* q_{23}^*(\mathrm{gr}\mathcal{Q})) \otimes \omega_Y) \otimes \omega_Z[-d_X - d_Y].$$

Now  $q_{12} \tilde{i}_{13} : T^*(X \times Y \times Z) \rightarrow T^*(X \times Y)$  is equal to  $(1 \times \iota) \pi_{12}$  and  $q_{23} \tilde{i}_{13} = \pi_{23}$  where  $\pi_{ij}$  are the natural projections. Also,  $\tilde{\pi}_{13} \tilde{q}_{13} = \pi_{13}$  and so we get

$$\begin{aligned} \vec{\mathrm{gr}}(\mathcal{Q} * \mathcal{P}) &\cong (1 \times \iota)^* \pi_{13*}(\pi_{12}^*(1 \times \iota)^*(\mathrm{gr}\mathcal{P}) \otimes \pi_{23}^*(\mathrm{gr}\mathcal{Q}) \otimes \omega_Y) \otimes \omega_Z[-d_X - d_Y] \\ &\cong \pi_{13*}((1 \times 1 \times \iota)^* \pi_{12}^*((1 \times \iota)^*(\mathrm{gr}\mathcal{P}) \otimes \omega_Y[-d_X]) \otimes (1 \times 1 \times \iota)^* \pi_{23}^*(\mathrm{gr}\mathcal{Q}) \otimes \omega_Z[-d_Y]) \\ &\cong \pi_{13*}(\pi_{12}^*((1 \times \iota)^*(\mathrm{gr}\mathcal{P}) \otimes \omega_Y[-d_X]) \otimes \pi_{23}^*((1 \times \iota)^*(\mathrm{gr}\mathcal{Q}) \otimes \omega_Z[-d_Y])) \\ &\cong \pi_{13*}(\pi_{12}^*(\vec{\mathrm{gr}}\mathcal{P}) \otimes \pi_{23}^*(\vec{\mathrm{gr}}\mathcal{Q})) \\ &\cong \vec{\mathrm{gr}}(\mathcal{Q}) * \vec{\mathrm{gr}}(\mathcal{P}). \end{aligned}$$

□

**4.4. Adjunction for  $\mathcal{D}_{X,h}$ -modules.** The Verdier duality functor  $\mathbb{D} : D(\mathcal{D}_{X,h}\text{-mod}) \rightarrow D(\mathcal{D}_{X,h}\text{-mod})^{\mathrm{opp}}$  is defined by

$$(7) \quad \mathbb{D}_X M = R\mathrm{Hom}_{\mathcal{D}_{X,h}}(M, \mathcal{D}_{X,h}) \otimes_{\mathcal{O}_{X,h}} \omega_{X,h}^{-1}[\dim(X)].$$

**Lemma 4.3.** *We have  $\mathbb{D}_X \mathcal{O}_{X,h} \cong \mathcal{O}_{X,h}\{2\dim(X)\}$ .*

*Proof.* We sketch the proof by following [HTT, Ex. 2.6.10]. The key is the locally free resolution

$$0 \rightarrow \mathcal{D}_{X,h} \otimes_{\mathcal{O}_{X,h}} \bigwedge^{d_X} \Theta_{X,h}\{-2d_X\} \rightarrow \cdots \rightarrow \mathcal{D}_{X,h} \otimes_{\mathcal{O}_{X,h}} \Theta_{X,h}\{-2\} \rightarrow \mathcal{D}_{X,h} \rightarrow \mathcal{O}_{X,h} \rightarrow 0$$

where  $d_X := \dim(X)$ . Then proceeding as in [HTT] we find that applying  $R\mathrm{Hom}_{\mathcal{D}_{X,h}}(\cdot, \mathcal{D}_{X,h})$  leaves us with

$$\cdots \rightarrow \mathrm{Hom}_{\mathcal{O}_{X,h}}(\mathcal{O}_{X,h}, \Omega_{X,h}^{d_X-1} \otimes_{\mathcal{O}_{X,h}} \mathcal{D}_{X,h}\{2d_X-2\}) \xrightarrow{\delta} \mathrm{Hom}_{\mathcal{O}_{X,h}}(\mathcal{O}_{X,h}, \Omega_{X,h}^{d_X} \otimes_{\mathcal{O}_{X,h}} \mathcal{D}_{X,h}\{2d_X\}).$$

The cokernel is  $\mathrm{Hom}_{\mathcal{O}_{X,h}}(\mathcal{O}_{X,h}, \Omega_{X,h}^{d_X}\{2d_X\})$  as right  $\mathcal{D}_{X,h}$ -modules. Then passing to left  $\mathcal{D}_{X,h}$ -modules by tensoring with  $\omega_{X,h}^{-1}[d_X]$  we get the result.  $\square$

**Lemma 4.4.** *If  $f : X \rightarrow Y$  is a proper morphism then  $\int_f \circ \mathbb{D}_X \cong \mathbb{D}_Y \circ \int_f$ . Moreover, if  $f$  is an inclusion, then  $\mathbb{D}_Y(\delta_X) = \delta_X$ .*

*Proof.* The first assertion follows from the proof of [HTT, Thm. 2.7.2]. The second assertion follows from the first and the previous Lemma.  $\square$

**4.5. Adjoint functors and associated graded.** Fix smooth, proper varieties  $X, Y$  and consider a kernel  $\mathcal{P} \in D(\mathcal{D}_{X \times Y, h}\text{-mod})$ . As discussed in section 4.2 this induces a functor  $\Phi_{\mathcal{P}} : D(\mathcal{D}_{X,h}\text{-mod}) \rightarrow D(\mathcal{D}_{Y,h}\text{-mod})$ . The following result describes the kernels which induce the left and right adjoints of  $\Phi_{\mathcal{P}}$ .

**Proposition 4.5.** *Given  $\mathcal{P} \in D(\mathcal{D}_{X \times Y, h}\text{-mod})$  as above, we have  $\Phi_{\mathcal{P}}^L \cong \Phi_{\mathcal{P}_L}$  and  $\Phi_{\mathcal{P}}^R \cong \Phi_{\mathcal{P}_R}$  where  $\mathcal{P}_L := \mathbb{D}_{X \times Y}(\mathcal{P})\langle 2\dim(Y) \rangle$  and  $\mathcal{P}_R := \mathbb{D}_{X \times Y}(\mathcal{P})\langle 2\dim(X) \rangle$ .*

*Proof.* This follows by [G, Thm. 1.3.4].  $\square$

**Proposition 4.6.** *For  $M \in D(\mathcal{D}_{X,h}\text{-mod})$  we have*

$$\mathrm{gr}(\mathbb{D}_X M) \cong \iota^* R\mathrm{Hom}_{\mathcal{O}_{T^*X}}(\mathrm{gr}(M), \mathcal{O}_{T^*X}) \otimes \omega_X^{-1}[\dim(X)]$$

where  $\omega_X$  is the canonical bundle of  $X$  pulled back to  $T^*X$  and  $\iota$  acts by  $(-1)$  on the fibers of  $T^*X$ .

*Proof.* This result follows directly from the definition of  $\mathbb{D}_X$  from (7). The only subtlety is the appearance of  $\iota$ . This is because the right action of  $\mathcal{D}_X$  on  $\omega_X$  is by  $-\mathrm{Lie}\theta$  which means that locally

$$\partial_i \cdot (f dz_1 \wedge \cdots \wedge dz_n) = -(\partial_i f) dz_1 \wedge \cdots \wedge dz_n.$$

$\square$

**Corollary 4.7.** *Given  $\mathcal{P} \in D(\mathcal{D}_{X \times Y, h}\text{-mod})$  we have  $\mathrm{gr}(\mathcal{P}_L) \cong \mathrm{gr}(\mathcal{P})_L$  and  $\mathrm{gr}(\mathcal{P}_R) \cong \mathrm{gr}(\mathcal{P})_R$ .*

*Proof.* Inside  $D(\mathcal{O}_{T^*Y \times T^*X}\text{-mod})$  we have the following

$$\begin{aligned} \mathrm{gr}(\mathcal{P}_L) &\cong \mathrm{gr}(\mathbb{D}_{X \times Y}(\mathcal{P}))\langle 2d_Y \rangle \\ &\cong (1 \times \iota)^* \mathrm{gr}(\mathbb{D}_{X \times Y}(\mathcal{P})) \otimes \omega_X[-d_X]\langle 2d_Y \rangle \\ &\cong (1 \times \iota)^*(\iota \times \iota)^*(\mathrm{gr}(\mathcal{P}))^\vee \otimes \omega_{X \times Y}^{-1} \otimes \omega_X[\dim(X \times Y)][-d_X]\langle 2d_Y \rangle \\ &\cong (\iota \times 1)^* \mathrm{gr}(\mathcal{P})^\vee \otimes \omega_Y^{-1}[d_Y]\langle 2d_Y \rangle \\ &\cong ((\iota \times 1)^* \mathrm{gr}(\mathcal{P}) \otimes \omega_Y[-d_Y])^\vee \langle 2d_Y \rangle \\ &\cong \mathrm{gr}(\mathcal{P})^\vee \otimes \omega_{T^*Y}[2d_Y]\{-2d_Y\} \\ &\cong \mathrm{gr}(\mathcal{P})_L \end{aligned}$$

where we write  $d_X = \dim(X)$  and  $d_Y = \dim(Y)$ . Here we used Proposition 4.6 to obtain the third line and  $\omega_{T^*Y} \cong \mathcal{O}_{T^*Y}\{2d_Y\}$  to obtain the second last line. This proves the first assertion. The fact that  $\tilde{\mathrm{gr}}(\mathcal{P}_R) \cong \tilde{\mathrm{gr}}(\mathcal{P})_R$  follows similarly.  $\square$

## 5. MIXED HODGE MODULES

We will need some results from Saito's theory of pure and mixed Hodge modules [Sa1, Sa2] which we now recall. For any smooth variety  $X$ , Saito constructed abelian categories of polarizable pure Hodge modules of weight  $w$ ,  $\mathrm{HM}(X, w)$ , and polarizable mixed Hodge modules,  $\mathrm{MHM}(X)$ . We have an exact forgetful functor to filtered  $\mathcal{D}_X$ -modules and subsequently a functor  $\mathbf{G} : \mathrm{MHM}(X) \rightarrow \mathcal{D}_{X,h}\text{-mod}$ . This functor is exact because for any morphism in  $\mathrm{MHM}$ , the associated morphism of filtered  $\mathcal{D}$ -modules is strict with respect to the filtration. We denote by  $\{1\} : \mathrm{MHM}(X) \rightarrow \mathrm{MHM}(X)$  the functor which shifts the filtration. This implies that  $\mathbf{G}(M\{1\}) = \mathbf{G}(M)\{1\}$  for any  $M \in \mathrm{MHM}(X)$ .

An object  $M \in \mathrm{MHM}(X)$  is equipped with an increasing filtration, denoted  $W_i(M)$ , called the *weight filtration*. The subquotients  $\mathrm{gr}_i^W(M) = W_i(M)/W_{i-1}(M)$  lie in the category  $\mathrm{HM}(X)$ .

Saito shows in [Sa2] that for any morphism  $f : X \rightarrow Y$  there exist associated functors  $f_* : D(\mathrm{MHM}(X)) \rightarrow D(\mathrm{MHM}(Y))$  and  $f^! : D(\mathrm{MHM}(Y)) \rightarrow D(\mathrm{MHM}(X))$ .

The functor  $f_*$  is compatible with the push-forward of  $\mathcal{D}_{X,h}$ -modules for proper  $f$  and is compatible with the push-forward of  $\mathcal{D}_X$ -modules for any  $f$  (the latter result follows immediately from the definitions).

**Theorem 5.1.** [Sa2, Thm. 4.3] *If  $f : X \rightarrow Y$  is proper, then*

$$\mathbf{G} \circ f_* = \int_f \circ \mathbf{G} : D(\mathrm{MHM}(X)) \rightarrow D(\mathcal{D}_{Y,h}\text{-mod}).$$

**Proposition 5.2.** *If  $f : X \rightarrow Y$  is any morphism, then for  $M \in D(\mathrm{MHM}(X))$ ,*

$$\mathbf{G}(f_*(M)) \otimes_{\mathbb{C}[h]} \mathbb{C}_1 = \int_f \mathbf{G}(M) \otimes_{\mathbb{C}[h]} \mathbb{C}_1$$

**Remark 5.3.** The pullback  $f^!$  of mixed Hodge modules is similarly compatible with the pullback  $f^\dagger$  of  $\mathcal{D}$ -modules (though we will not directly use this fact in the paper).

**5.1. Structure theorem and decomposition theorem.** An object  $M \in \mathrm{HM}(X, w)$  has strict support  $Z \subseteq X$  if it has no non-zero sub or quotient object supported on proper closed subvarieties of  $Z$ . From the definition of  $\mathrm{HM}(X, w)$ , we see that every object  $M \in \mathrm{HM}(X, w)$  admits a decomposition  $M = \bigoplus_{Z \subseteq X} M_Z$ , where  $M_Z$  has strict support on  $Z$  (see [Sc, section 12]).

The following structure theorem appears in [Sa2] (see [Sc, Theorem 15.1]).

**Theorem 5.4.** *Let  $\mathcal{V}$  be a variation of Hodge structure of weight  $w$  on a smooth open subset  $U \subset Z$ . Then  $\mathcal{V}$  extends uniquely to a Hodge module on  $X$  of weight  $w + \dim(Z)$  with strict support  $Z$ . Moreover, every Hodge module with strict support  $Z$  is obtained this way.*

In particular, if we start with the constant rank 1 variation of Hodge structure on an open subset of  $Z$  of weight  $-\dim(Z)$ , we obtain a Hodge module on  $Z$  which we denote by  $\mathrm{IC}_{Z,m}$ .



We define  $\mathrm{IC}_{Z,h} = \mathbf{G}(\mathrm{IC}_{Z,m})$ . In the case, when  $Z$  is smooth, we will write  $\delta_{Z,m}$  for  $\mathrm{IC}_{Z,m}$  — note that in this case  $\mathrm{IC}_{Z,h} = \delta_{Z,h}$  so the notation is consistent.

Saito also proved the following decomposition theorem for projective morphisms.

**Theorem 5.5.** [Sa1, Section 5] *If  $M \in \mathrm{HM}(X, w)$  and  $f : X \rightarrow Y$  is projective, then each  $H^i(f_*(M))$  lies in  $\mathrm{HM}(Y, w + i)$ . Moreover,  $f_*(M) = \bigoplus_i H^i(f_*(M))[-i]$ .*

**Corollary 5.6.** *If  $f : X \rightarrow Y$  is a small resolution then  $f_*(\delta_{X,m}) \cong \mathrm{IC}_{Y,m}$ .*

*Proof.* It suffices to show that the left hand side is indecomposable (because of Theorems 5.4 and 5.5). Suppose it is decomposable. Then  $\mathbf{G}(f_*\mathcal{O}_{X,m}) \otimes_{\mathbb{C}[h]} \mathbb{C}_1$  is decomposable (here we use that  $\mathbf{G}(\cdot)$  is  $\mathbb{C}[h]$ -torsion free). But  $\mathbf{G}(f_*\mathcal{O}_{X,m}) \otimes_{\mathbb{C}[h]} \mathbb{C}_1 \cong \int_f \mathcal{O}_X$  by Proposition 5.2. The right hand side is just  $\mathrm{IC}_Y$  which is indecomposable (contradiction).  $\square$

**5.2. Kashiwara's theorem and base change.** Finally, we recall the base change theorem in the category  $\mathrm{MHM}$ . If  $i : Y \rightarrow X$  is an inclusion of smooth varieties then [Sa2, Formula 4.24] states that  $i_* : \mathrm{MHM}(Y) \rightarrow \mathrm{MHM}_Y(X)$  is an equivalence of categories, where the right hand side denotes the full subcategory of  $\mathrm{MHM}(X)$  consisting of objects which are set-theoretically supported on  $Y$ .

**Theorem 5.7.** *If  $i : Y \rightarrow X$  is an inclusion of smooth varieties then the functor  $i^* = i^!$ , when restricted to  $\mathrm{MHM}_Y(X)$ , is the inverse to  $i_*$ .*

*Proof.* Let  $j : U \subset X$  denote the complement to  $Y$ . According to [Sa2], section 4.4, we have an exact triangle for any  $M$  in  $D^b(\mathrm{MHM}(X))$

$$j!j^!(M) \rightarrow M \rightarrow i_*i^*M$$

which implies that the natural map  $M \rightarrow i_*i^*M$  is an isomorphism for any  $M \in \mathrm{MHM}_Y(X)$ . Now, if we take any  $N$  in  $\mathrm{MHM}(Y)$ , the above implies that we have an isomorphism

$$i_*i^*i_*(N) \rightarrow i_*(N)$$

which means, since  $i_*$  is fully faithful, that the natural map  $i^*i_*(N) \rightarrow N$  is an isomorphism. This proves the assertion for  $i^*$ . The proof for  $i^!$  is the same but with the above exact triangle replaced by its dual

$$i_*i^!(M) \rightarrow M \rightarrow j_*j^!(M).$$

$\square$

**Remark 5.8.** The theorem above holds for  $\mathcal{D}$ -modules (this is Kashiwara's theorem) but does not hold for  $\mathcal{D}_h$ -modules. One reason is that (in general) the analogue of Theorem 5.1 does not hold for pullbacks (meaning that pullback does not commute with  $\mathbf{G}$ ).

**Theorem 5.9.** *In the fiber product diagram below we have  $g^!f_* \cong f'_*(g')^!$ .*

$$\begin{array}{ccc} Y \times_X Z & \xrightarrow{g'} & Y \\ f' \downarrow & & \downarrow f \\ Z & \xrightarrow{g} & X \end{array}$$

*Proof.* The proof of this is the same as for  $\mathcal{D}$ -modules (see [HTT, Sec. 1.7]) and relies only on Kashiwara's theorem (Theorem 5.7) and the adjunction triangles for pushforwards and pullbacks.  $\square$

6. THE  $\mathfrak{sl}_2$  ACTION ON  $\mathcal{D}_h$ -MODULES

**6.1. The action.** We will now define a categorical  $\mathfrak{sl}_2$  action. Fix an integer  $N$  and define a 2-category  $\mathcal{K}_{\mathcal{D}}$  with objects  $\lambda = -N, -N+2, \dots, N-2, N$  and 1-morphism categories  $\mathcal{K}_{\mathcal{D}}(\lambda, \lambda') := D(\mathcal{D}_{\mathbb{G}(k,N) \times \mathbb{G}(k',N),h})$  where  $k, k'$  are related to  $\lambda, \lambda'$  by  $\lambda = N-2k, \lambda' = N-2k'$ . We will also write  $d_k := k(N-k) = \dim(\mathbb{G}(k, N))$ .

We have the following correspondence

$$\mathbb{G}(k, N) = \{0 \subset V \subset \mathbb{C}^N\} \xleftarrow{p_1} C(\lambda) := \{0 \subset V' \subset V \subset \mathbb{C}^N\} \xrightarrow{p_2} \{0 \subset V' \subset \mathbb{C}^N\} = \mathbb{G}(k-1, N)$$

where  $p_1$  and  $p_2$  forget  $V'$  and  $V$  respectively and  $\lambda = N-2k+1 = d_k - d_{k-1}$ . These can be used to define kernels

$$\begin{aligned} \mathcal{E}(\lambda) &:= \delta_{C(\lambda),h} \langle d_k \rangle \in D(\mathcal{D}_{\mathbb{G}(k,N) \times \mathbb{G}(k-1,N),h}\text{-mod}) \\ \mathcal{F}(\lambda) &:= \delta_{C(\lambda),h} \langle d_{k-1} \rangle \in D(\mathcal{D}_{\mathbb{G}(k-1,N) \times \mathbb{G}(k,N),h}\text{-mod}). \end{aligned}$$

**Lemma 6.1.** *The left and right adjoints are given by*

$$\mathcal{E}(\lambda)_L \cong \mathcal{F}(\lambda) \langle -\lambda \rangle \quad \text{and} \quad \mathcal{E}(\lambda)_R \cong \mathcal{F}(\lambda) \langle \lambda \rangle.$$

*Proof.* Using Proposition 4.5 we have

$$\mathcal{E}(\lambda)_L \cong \mathbb{D}(\delta_{C(\lambda),h}) \langle -d_k \rangle \langle 2d_{k-1} \rangle \cong \delta_{C(\lambda),h} \langle d_{k-1} \rangle \langle -\lambda \rangle = \mathcal{F}(\lambda) \langle -\lambda \rangle.$$

A similar argument also computes  $\mathcal{E}(\lambda)_R$ . □

**6.2. Proof of the commutation relation.** To prove the main  $\mathfrak{sl}_2$  commutation relation we first need the following fact about small resolutions of  $GL_N$  orbit closures in products of Grassmannians. This is closely related to a result of Zelevinsky [Ze]. Fix two integers  $k, l$  with  $k \leq l$ . The  $GL_N$  orbits in  $\mathbb{G}(k, N) \times \mathbb{G}(l, N)$  are the strata  $Z_s := \{(V, V') : \dim V \cap V' = s\}$ , where  $s = \max(0, k+l-N), \dots, k$ . We have  $\overline{Z}_s = Z_s \cup \dots \cup Z_k$ . One can define a resolution  $\pi : P_s \rightarrow \overline{Z}_s$  where

$$P_s := \{(V, V', V'') \in \mathbb{G}(k, N) \times \mathbb{G}(l, N) \times \mathbb{G}(s, N) : V'' \subset V \cap V'\}.$$

and  $\pi$  forgets  $V''$ .

**Proposition 6.2.** *If  $k+l \leq N$  then the map  $\pi : P_s \rightarrow \overline{Z}_s$  is a small resolution.*

*Proof.* Since  $P_s$  is an iterated bundle of Grassmannians it must be smooth. It is clear that  $\pi$  is one-to-one over  $Z_s$  which means that  $\pi$  is a resolution. It remains to show that  $\pi$  is small.

The relevant strata are the  $Z_t \subset \overline{Z}_s$  for  $t > s$ . An elementary computation (based on  $\dim(Z_s) = \dim(P_s)$  and the description of  $P_s$  as an iterated Grassmannian bundle) shows that

$$\dim(Z_s) - \dim(Z_t) = (t-s)(s+t+N-k-l).$$

On the other hand, for a point  $(V, V') \in Z_t$ , we see that  $\pi^{-1}(V, V')$  is the Grassmannian of  $s$ -dimensional subspaces of  $V \cap V'$ . Since  $\dim V \cap V' = t$ , we see that  $\dim \pi^{-1}(V, V') = s(t-s)$ . Since  $k+l \leq N$  and  $t < s$ , we see that  $2s < s+t+N-k-l$  and conclude that

$$2\dim \pi^{-1}(V, V') < \dim(Z_s) - \dim(Z_t)$$

which proves that  $\pi$  is small. □

**Proposition 6.3.** *We have the following relations*

- (i)  $\mathcal{F}(\lambda + 1) * \mathcal{E}(\lambda + 1) \cong \mathcal{E}(\lambda - 1) * \mathcal{F}(\lambda - 1) \bigoplus_{[-\lambda]} \delta_{\Delta, h} \langle d_k \rangle$  if  $\lambda \leq 0$ ,  
(ii)  $\mathcal{E}(\lambda - 1) * \mathcal{F}(\lambda - 1) \cong \mathcal{F}(\lambda + 1) * \mathcal{E}(\lambda + 1) \bigoplus_{[\lambda]} \delta_{\Delta, h} \langle d_k \rangle$  if  $\lambda \geq 0$

inside  $D(\mathcal{D}_{\mathbb{G}(k, N) \times \mathbb{G}(k, N), h}\text{-mod})$ , where  $\lambda = N - 2k$ .

*Proof.* We will prove that case  $\lambda \geq 0$  (the case  $\lambda \leq 0$  is the same). First, we have

$$\begin{aligned} \mathcal{F}(\lambda + 1) * \mathcal{E}(\lambda + 1) &\cong \int_{p_{13}} p_{12}^\dagger \delta_{C(\lambda+1), h} \otimes p_{23}^\dagger \delta_{C(\lambda+1), h} \langle d_k + d_{k-1} \rangle \\ &\cong \int_{p_{13}} \delta_{p_{12}^{-1}(C(\lambda+1)), h} \langle d_k \rangle \otimes \delta_{p_{23}^{-1}(C(\lambda+1)), h} \langle d_k \rangle \langle d_k + d_{k-1} \rangle \\ &\cong \int_{p_{13}} \delta_{P_{k-1}, h} \langle d_k \rangle \end{aligned}$$

where  $P_{k-1} = \{0 \subset V' \subset V, V'' \subset \mathbb{C}^N\}$  with  $\dim(V') = k - 1, \dim(V/V') = \dim(V''/V') = 1$ . Note that the third isomorphism follows from Corollary 3.7 and a dimension count (the tensor product in the second line is happening in the triple product which has dimension  $2d_k + d_{k-1}$ ).

Since  $p_{13}$  forgets  $V'$  the image  $p_{13}(P_{k-1})$  is equal to  $\overline{Z_{k-1}} = \{0 \subset V, V'' \subset \mathbb{C}^N : \dim(V \cap V'') \geq k - 1\}$ . By Proposition 6.2 the map  $\pi : P_{k-1} \rightarrow \overline{Z_{k-1}}$  is small. Subsequently, by Corollary 5.6 we get  $\int_{p_{13}} \delta_{P_{k-1}, h} = \text{IC}_{\overline{Z_{k-1}}, h}$  and thus  $\mathcal{F}(\lambda + 1) * \mathcal{E}(\lambda + 1) \cong \text{IC}_{\overline{Z_{k-1}}, h} \langle d_k \rangle$ .

On the other hand, a similar argument shows

$$\mathcal{E}(\lambda - 1) * \mathcal{F}(\lambda - 1) \cong \int_{p_{13}} \delta_{P', h} \langle d_k \rangle$$

where  $P' := \{0 \subset V, V'' \subset V' \subset \mathbb{C}^N\}$  with  $\dim(V') = k + 1, \dim(V'/V) = \dim(V'/V'') = 1$ . The pushforward  $\int_{p_{13}} \delta_{P', h}$  (which is now more difficult to calculate because the map is no longer small) is computed in Lemma 6.4. The result follows.  $\square$

**Lemma 6.4.** *Let  $P' = \{0 \subset V, V'' \subset V' \subset \mathbb{C}^N\} \subset \mathbb{G}(k, N) \times \mathbb{G}(k + 1, N) \times \mathbb{G}(k, N)$  and consider the projection  $p_{13}$  which forgets  $V'$  as in the proof of Proposition 6.3. Then*

$$\int_{p_{13}} \delta_{P', h} \cong \text{IC}_{\overline{Z_{k-1}}, h} \bigoplus_{[N-2k]} \int_{\Delta} \delta_{\mathbb{G}(k, N), h}.$$

*Proof.* We will show the corresponding result

$$(8) \quad p_{13*}(\delta_{P', m}) \cong \text{IC}_{\overline{Z_{k-1}}, m} \bigoplus_{[N-2k]} \Delta_*(\delta_{\mathbb{G}(k, N), m})$$

at the level of MHM. The result then follows by applying  $\mathbf{G}$  and using that  $\mathbf{G}$  commutes with proper pushforward.

The decomposition theorem for Hodge modules tells us that

$$(9) \quad p_{13*}(\delta_{P', m}) \cong \bigoplus_i H^i(p_{13*}(\delta_{P', m}))[-i]$$

and the structure theorem tells us that

$$(10) \quad H^i(p_{13*}(\delta_{P', m})) = N_i \oplus D_i$$

where  $N_i$  has strict support  $\overline{Z_{k-1}}$  and  $D_i$  has strict support the diagonal  $\mathbb{G}(k, N)$  (these are the only possible strict supports because the whole situation is  $GL_N$  equivariant and the image of  $p_{13}$  is  $\overline{Z_{k-1}}$ ).

Since  $p_{13*}$  is an isomorphism over  $Z_{k-1}$ , the structure theorem for Hodge modules shows us that  $N_i = 0$  for  $i \neq 0$  and that  $N_0 = \mathrm{IC}_{\overline{Z_{k-1}}, m}$ . Thus we conclude that

$$p_{13*}(\delta_{P', m}) = \mathrm{IC}_{\overline{Z_{k-1}}, m} \oplus_i D_i[-i]$$

where each  $D_i$  has strict support along the diagonal. We will now consider the base change over the diagonal. Using the following commutative diagram

$$\begin{array}{ccccc} Q := \{0 \subset V = V'' \subset V' \subset \mathbb{C}^N\} & \xrightarrow{\tilde{i}} & \mathbb{G}(k, N) \times \mathbb{G}(k+1, N) & \xrightarrow{\tilde{p}_{13}} & \mathbb{G}(k, N) \\ \downarrow \hat{\Delta} & & \downarrow \tilde{\Delta} & & \downarrow \Delta \\ P' & \xrightarrow{i} & \mathbb{G}(k, N) \times \mathbb{G}(k+1, N) \times \mathbb{G}(k, N) & \xrightarrow{p_{13}} & \mathbb{G}(k, N) \times \mathbb{G}(k, N) \end{array}$$

together with the base change formula we have

$$\begin{aligned} \Delta^! p_{13*} i_* \delta_{P', m} &\cong \tilde{p}_{13*} \tilde{\Delta}^! i_* \delta_{P', m} \cong (\tilde{p}_{13})_* \tilde{i}_* \hat{\Delta}^! \delta_{P', m} \\ &\cong \tilde{p}_{13*} \tilde{i}_* \delta_{Q, m} \langle -k \rangle \\ &\cong \bigoplus_{[N-k]} \delta_{\mathbb{G}(k, N), m} \langle -k \rangle. \end{aligned}$$

where the third isomorphism follows from the Hodge module analogue of Corollary 3.8 and the last isomorphism uses the fact that  $\tilde{p}_{13} \circ \tilde{i} : Q \rightarrow \mathbb{G}(k, N)$  is a  $\mathbb{P}^{N-k-1}$  fibration and gives a constant variation of Hodge structure with fibre  $H^*(\mathbb{P}^{N-k-1})$ . Note that as a Hodge structure,

$$H^*(\mathbb{P}^r) = \mathbb{C}[-r]\{r\} \oplus \mathbb{C}[2-r]\{r-2\} \cdots \oplus \mathbb{C}[r]\{-r\} = \bigoplus_{[r+1]} \mathbb{C}$$

A similar analysis shows that  $\Delta^! p_{13*} i_* \delta_{P_{k-1}, m} \cong \bigoplus_{[k]} \delta_{\mathbb{G}(k, N), m} \langle -N+k \rangle$ . Since  $p_{13*} \delta_{P_{k-1}, m} = \mathrm{IC}_{\overline{Z_{k-1}}, m}$  we also have

$$\Delta^! \mathrm{IC}_{\overline{Z_{k-1}}, m} \cong \bigoplus_{[k]} \delta_{\mathbb{G}(k, N), m} \langle -N+k \rangle.$$

Thus, we conclude that

$$\bigoplus_{[N-k]} \delta_{\mathbb{G}(k, N), m} \langle -k \rangle = \bigoplus_{[k]} \delta_{\mathbb{G}(k, N), m} \langle -N+k \rangle \oplus_i D_i[-i]$$

where each  $D_i$  has strict support along the diagonal. Thus we conclude that

$$\bigoplus_i D_i[-i] = \bigoplus_{[N-2k]} \delta_{\mathbb{G}(k, N), m}$$

The relation in (8) now follows.  $\square$

**6.3. Divided powers.** It is interesting in this case to identify explicitly the divided powers  $\mathcal{E}^{(r)}$  and  $\mathcal{F}^{(r)}$  for  $r \in \mathbb{N}$ . Consider the following correspondences

$$\mathbb{G}(k, N) = \{0 \subset V \subset \mathbb{C}^N\} \xleftarrow{p_1} C^r(\lambda) := \{0 \subset V' \subset V \subset \mathbb{C}^N\} \xrightarrow{p_2} \{0 \subset V' \subset \mathbb{C}^N\} = \mathbb{G}(k-r, N)$$

where  $\lambda = N - 2k + r$  (these generalize  $C(\lambda)$  from above). It turns out that

$$\begin{aligned} \mathcal{E}^{(r)}(\lambda) &:= \delta_{C^r(\lambda), h} \langle d_k \rangle \in D(\mathcal{D}_{\mathbb{G}(k, N) \times \mathbb{G}(k-r, N), h} \text{-mod}) \\ \mathcal{F}^{(r)}(\lambda) &:= \delta_{C^r(\lambda), h} \langle d_{k-r} \rangle \in D(\mathcal{D}_{\mathbb{G}(k-r, N) \times \mathbb{G}(k, N), h} \text{-mod}). \end{aligned}$$

By a similar argument as in the last section we have

$$\mathcal{E}^{(r_2)} * \mathcal{E}^{(r_1)} \cong \int_{p_{13}} \delta_{U, h} \langle d_k \rangle$$

where  $U := \{0 \subset V'' \subset V' \subset V \subset \mathbb{C}^N\}$  with  $\dim(V) = k + r_1$ ,  $\dim(V/V') = r_1$ ,  $\dim(V'/V'') = r_2$ . Since  $p_{13}$  forgets  $V'$  we have  $p_{13}(U) = C^{r_1+r_2}$  where the projection  $U \rightarrow p_{13}(U)$  is a  $\mathbb{G}(r_1, r_1 + r_2)$ -bundle. Thus

$$\int_{p_{13}} \delta_{U, h} \cong \bigoplus_{\begin{bmatrix} r_1+r_2 \\ r_1 \end{bmatrix}} \delta_{C^{r_1+r_2}, h}$$

since, as a Hodge structure,

$$H^*(\mathbb{G}(r_1, r_1 + r_2)) = \bigoplus_{\begin{bmatrix} r_1+r_2 \\ r_1 \end{bmatrix}} \mathbb{C}.$$

Thus we end up with

$$\mathcal{E}^{(r_2)} * \mathcal{E}^{(r_1)} \cong \bigoplus_{\begin{bmatrix} r_1+r_2 \\ r_1 \end{bmatrix}} \delta_{C^{r_1+r_2}, h} \langle d_k \rangle \cong \bigoplus_{\begin{bmatrix} r_1+r_2 \\ r_1 \end{bmatrix}} \mathcal{E}^{(r_1+r_2)}.$$

**6.4. Action of the nil affine Hecke algebra.** At the level of constructible sheaves this action is well known and relatively straight-forward to check. Via the Riemann-Hilbert correspondence one also gets an action in the context of  $\mathcal{D}$ -modules. We would like to lift this to an action on  $\mathcal{D}_h$ -modules.

First, let us explain how to lift the action of  $X(\lambda)$ . Recall (section 14.3.3 [PS]) that on a smooth projective complex variety equipped with a line bundle  $\mathcal{L}$  there is a Lefschetz operator  $L_{\mathcal{L}}$  which acts on any object in the derived category of filtered  $\mathcal{D}$ -modules (and hence of  $\mathcal{D}_h$ -modules). More specifically, for any  $\mathcal{D}_h$ -module  $M$  we have the operator

$$L_{\mathcal{L}} : M \rightarrow M[2]\{-2\}$$

Now, the variety used to define  $\mathcal{E}(\lambda)$ , namely  $C(\lambda) = \{0 \subset V' \subset V \subset \mathbb{C}^N\}$ , has the natural line bundle  $V/V'$ . Thus the Lefschetz operator associated to this line bundle yields a morphism

$$\delta_{C(\lambda), h} \rightarrow \delta_{C(\lambda), h}[2]\{-2\}$$

and the resulting action on the functor  $\mathcal{E}(\lambda)$  defines  $X(\lambda)$ .

One can also define  $T(\lambda)$  in a similar manner. We then have to show the nil affine Hecke relations such as  $(X(\lambda + 1)I) \circ T(\lambda) - T(\lambda) \circ (IX(\lambda - 1)) - I = 0$ . Since the left hand side

is zero for  $\mathcal{D}$ -modules we know that it is equal to  $\mathbb{C}[h]$ -torsion inside  $\text{End}^2(\mathcal{E}(\lambda+1)\mathcal{E}(\lambda-1))$ . Unfortunately such  $\mathbb{C}[h]$ -torsion exists so we cannot immediately conclude that the nilHecke relations hold.

It might be possible to follow the usual proof to show that the nilHecke relations hold on the nose (not just modulo  $\mathbb{C}[h]$ -torsion). However, since this is somewhat involved, we will give an alternative proof based on [C2]. There we showed that to obtain the action of the nil affine Hecke algebra, it suffices to define  $\theta(\lambda) \in \text{End}^2(\text{Id}_\lambda)$  such that

$$I\theta(\lambda)I \in \text{Hom}(\mathcal{E}(\lambda+1)\text{Id}_\lambda\mathcal{E}(\lambda-1), \mathcal{E}(\lambda+1)\text{Id}_\lambda\mathcal{E}(\lambda-1)[2]\{-2\})$$

induces an isomorphism between the two summands  $\mathcal{E}^{(2)}[1]\{-1\}$  on either side.

In our setup we define  $\theta(\lambda)$  to be just  $L_{\mathcal{L}}$  where  $\mathcal{L}$  is the line bundle  $\det(V)$  on  $\mathbb{G}(k, N) = \{0 \subset V \subset \mathbb{C}^N\}$ . At the level of  $\mathcal{D}$ -modules (or constructible sheaves) it is easy to check that  $L_{\mathcal{L}} - a \cdot \text{Id}$  induces zero between the two summands (for some  $a \in \mathbb{C}$ ). This essentially follows from the fact that the nilHecke relations holds for constructible sheaves. Thus  $L_{\mathcal{L}} - a \cdot \text{Id}$  is equal to  $\mathbb{C}[h]$ -torsion inside  $\text{End}(\mathcal{E}^{(2)}[1]\{-1\})$ . However, in this case there is no  $\mathbb{C}[h]$ -torsion (this is just a homological degree zero endomorphism) so we are done.

## 7. ASSOCIATED GRADED OF THE $\mathfrak{sl}_2$ ACTION

Consider the 2-category  $\mathcal{K}_{\mathcal{O}}$  with the same objects as  $\mathcal{K}_{\mathcal{D}}$  but morphism categories

$$\mathcal{K}_{\mathcal{O}}(\lambda, \lambda') = D(\mathcal{O}_{T^*\mathbb{G}(k, N) \times T^*\mathbb{G}(k', N)\text{-mod}}^{\mathbb{C}^\times})$$

where  $\lambda = N - 2k, \lambda' = N - 2k'$  as before. We have a functor

$$\vec{\text{gr}} : D(\mathcal{D}_{\mathbb{G}(k, N) \times \mathbb{G}(k', N), h}\text{-mod}) \rightarrow D(\mathcal{O}_{T^*\mathbb{G}(k, N) \times T^*\mathbb{G}(k', N)\text{-mod}}^{\mathbb{C}^\times})$$

and, by Proposition 4.2, these functors combine to give an associated graded 2-functor  $\text{gr} : \mathcal{K}_{\mathcal{D}} \rightarrow \mathcal{K}_{\mathcal{O}}$ . Thus, the categorical  $\mathfrak{sl}_2$ -action defined in the previous section with target  $\mathcal{K}_{\mathcal{D}}$  gives rise to a categorical  $\mathfrak{sl}_2$ -action with target  $\mathcal{K}_{\mathcal{O}}$ . In this section, we will compute this action explicitly.

We will compute  $\vec{\text{gr}}(\delta_{C^r(\lambda), h})$  where  $C^r(\lambda)$  is the correspondence from section 6.3 (technically we only need the case  $r = 1$  but it is interesting to compute this for all  $r$ ). Recall that

$$T^*\mathbb{G}(k, N) = \{(X, V) : X \in \text{End}(\mathbb{C}^N), X(V) = 0, X(\mathbb{C}^N) \subset V\}$$

and likewise for  $T^*\mathbb{G}(k-r, N)$ . Moreover, the conormal bundle of  $C^r(\lambda) \subset \mathbb{G}(k, N) \times \mathbb{G}(k-r, N)$  can be identified with

$$\mathfrak{C}^r(\lambda) := \{(X, V, V') : 0 \subset V' \subset V \subset \mathbb{C}^N, X(\mathbb{C}^N) \subset V, X(V) \subset V', X(V') = 0\}.$$

**Proposition 7.1.** *As a kernel inside  $T^*\mathbb{G}(k, N) \times T^*\mathbb{G}(k-r, N)$  we have*

$$\vec{\text{gr}}(\delta_{C^r(\lambda), h}) \cong \mathcal{O}_{\mathfrak{C}^r(\lambda)} \otimes \det(V)^{-k+r} \otimes \det(V')^k[-d_k]\{r(k-r) + d_k\}$$

while, as a kernel inside  $T^*\mathbb{G}(k-r, N) \times T^*\mathbb{G}(k, N)$ , we have

$$\vec{\text{gr}}(\delta_{C^r(\lambda), h}) \cong \mathcal{O}_{\mathfrak{C}^r(\lambda)} \otimes \det(V'/V)^{N-k} \otimes \det(\mathbb{C}^N/V')^{-r}[-d_{k-r}]\{r(N-k) + d_{k-r}\}$$

using the usual convention that in the second case  $V'$  is the vector bundle on  $T^*\mathbb{G}(k, N)$ .

*Proof.* First we compute  $\mathrm{gr}(\delta_{C^r(\lambda),h})$ . Using Corollary 3.5 we find that

$$\mathrm{gr}(\delta_{C^r(\lambda),h}) \cong \mathcal{O}_{\mathfrak{C}^r(k,N)} \otimes \omega_{X/Y} \{\dim(C^r(\lambda))\}$$

where  $X = C^r(\lambda)$ ,  $Y = \mathbb{G}(k, N) \times \mathbb{G}(k-r, N)$  and  $\dim(C^r(\lambda)) = r(k-r) + d_k$ . Moreover, it is a standard exercise to calculate that

$$\begin{aligned} \omega_{\mathbb{G}(k,N)} &\cong \det(V)^N \otimes \det(\mathbb{C}^N)^{-k} \\ \omega_{\mathfrak{C}^r(k,N)} &\cong \det(V')^k \otimes \det(V)^{N-k+r} \otimes \det(\mathbb{C}^N)^{-k}. \end{aligned}$$

It then follows that

$$\begin{aligned} \mathrm{gr}(\delta_{C^r(\lambda),h}) &\cong \mathcal{O}_{\mathfrak{C}^r(\lambda)} \otimes \omega_{\mathfrak{C}^r(\lambda)} \otimes \omega_{\mathbb{G}(k,N) \times \mathbb{G}(k-r,N)}^{\vee} \{r(k-r) + d_k\} \\ &\cong \mathcal{O}_{\mathfrak{C}^r(\lambda)} \otimes \det(V)^{-k+r} \det(V')^{k-N} \det(\mathbb{C}^N)^{k-r} \{r(k-r) + d_k\}. \end{aligned}$$

Thus, as a kernel inside  $T^*\mathbb{G}(k, N) \times T^*\mathbb{G}(k-r, N)$  we have

$$\begin{aligned} \vec{\mathrm{gr}}(\delta_{C^r(\lambda),h}) &\cong \mathrm{gr}(\delta_{C^r(\lambda),h}) \otimes \omega_{\mathbb{G}(k-r,N)}[-d_k] \\ &\cong \mathcal{O}_{\mathfrak{C}^r(\lambda)} \otimes \det(V)^{-k+r} \det(V')^k[-d_k] \{r(k-r) + d_k\} \end{aligned}$$

while as a kernel inside  $T^*\mathbb{G}(k-r, N) \times T^*\mathbb{G}(k, N)$  we have

$$\begin{aligned} \vec{\mathrm{gr}}(\delta_{C^r(\lambda),h}) &\cong \mathrm{gr}(\delta_{C^r(\lambda),h}) \otimes \omega_{\mathbb{G}(k,N)}[-d_{k-r}] \\ &\cong \mathcal{O}_{\mathfrak{C}^r(\lambda)} \otimes \det(V)^{k-N} \det(V)^{N-k+r} \det(\mathbb{C}^N)^r[-d_{k-r}] \{r(N-k) + d_{k-r}\}. \end{aligned}$$

This agrees with what we wanted to show.  $\square$

**Corollary 7.2.** *The associated graded of the categorical  $\mathfrak{sl}_2$  action with target  $\mathcal{K}_{\mathcal{D}}$  is the action with target  $\mathcal{K}_{\mathcal{O}}$  and kernels*

$$\begin{aligned} \vec{\mathrm{gr}}(\mathcal{E}^{(r)}(\lambda)) &= \mathcal{O}_{\mathfrak{C}^r(\lambda)} \otimes \det(V)^{-k+r} \otimes \det(V')^k \{r(k-r)\} \in D(\mathcal{O}_{T^*\mathbb{G}(k,N) \times T^*\mathbb{G}(k-r,N)}\text{-mod}^{\mathbb{C}^\times}) \\ \vec{\mathrm{gr}}(\mathcal{F}^{(r)}(\lambda)) &= \mathcal{O}_{\mathfrak{C}^r(\lambda)} \otimes \det(V'/V)^{N-k} \otimes \det(\mathbb{C}^N/V')^{-r} \{r(N-k)\} \in D(\mathcal{O}_{T^*\mathbb{G}(k-r,N) \times T^*\mathbb{G}(k,N)}\text{-mod}^{\mathbb{C}^\times}) \end{aligned}$$

where  $\lambda = N - 2k + r$ .

**Remark 7.3.** In [CKL1] we defined another categorical  $\mathfrak{sl}_2$  action with target  $\mathcal{K}_{\mathcal{O}}$ . The kernels which induce this action are not exactly the same as those from Corollary 7.2 but the difference is only some line bundles which can essentially be accounted for by conjugation. On the other hand, the categorical  $\mathfrak{sl}_2$  action defined in [CK4] does agree with the one above, after restricting to the open subsets  $T^*\mathbb{G}(k, N) \subset Y(k, N-k)$  and matching up the notation (cf. Appendix A.2 of [CK4]).

## 8. THE EQUIVALENCE $\mathsf{T}$

In this section we study the equivalence

$$\mathsf{T} : D(\mathcal{D}_{\mathbb{G}(k,N),h}\text{-mod}) \rightarrow D(\mathcal{D}_{\mathbb{G}(N-k,N),h}\text{-mod})$$

induced by the categorical  $\mathfrak{sl}_2$  action from section 6. For notational simplicity we assume  $2k \leq N$ . For  $s = 0, \dots, k$  define

$$\Theta^s := \mathcal{F}^{(N-k-s)} * \mathcal{E}^{(k-s)} \in D(\mathcal{D}_{\mathbb{G}(k,N) \times \mathbb{G}(N-k,N),h}\text{-mod}).$$



From these we can define Rickard's complex  $\Theta^k\langle -k \rangle \rightarrow \cdots \rightarrow \Theta^1\langle -1 \rangle \rightarrow \Theta^0$  where the differential  $\Theta^s\langle -s \rangle \rightarrow \Theta^{s-1}\langle -s+1 \rangle$  is given by the composition

$$\mathcal{F}^{(N-k-s)} * \mathcal{E}^{(k-s)}\langle -s \rangle \rightarrow \mathcal{F}^{(N-k-s)} * \mathcal{F} * \mathcal{E}^{(k-s)}\langle N-3s+1 \rangle \rightarrow \mathcal{F}^{(N-k-s+1)} * \mathcal{E}^{(k-s+1)}\langle -s+1 \rangle$$

where the first arrow uses the adjunction and the second map consists of two projections. The following result follows from [CR, CKL1].

**Theorem 8.1.** *The complex  $\Theta^*$  has a unique right convolution  $\mathcal{T} := \text{Cone}(\Theta^*)$  which induces an equivalence  $\mathbb{T} : D(\mathcal{D}_{\mathbb{G}(k,N),h}\text{-mod}) \xrightarrow{\sim} D(\mathcal{D}_{\mathbb{G}(N-k,N),h}\text{-mod})$ .*

**Remark 8.2.** Convolutions of complexes in a triangulated category were introduced by Orlov [O]. They are defined by an iterated cone construction. We choose to work with right convolutions, meaning that we start the iterated cones on the right.

**Remark 8.3.** The complex  $\Theta^*$  is actually the dual of the complex we usually considered in (for instance) [CKL1]. The complex in those cases is  $\Theta^0\langle -k \rangle \rightarrow \cdots \rightarrow \Theta^{k-1}\langle -1 \rangle \rightarrow \Theta^k$ .

For  $s = 0, \dots, k$  recall the locus  $Z_s \subset \mathbb{G}(k, N) \times \mathbb{G}(N-k, N)$  consisting of pairs  $(V, V')$  such that  $\dim(V \cap V') = s$  (notice that  $Z_0$  is an open). Note that  $\dim(Z_s) = 2d_k - s^2$ . The following is the main result in this section.

**Theorem 8.4.** *We have  $\mathcal{T} \cong \mathbb{G}(j_*\delta_{Z_0,m})\langle d_k \rangle$  where  $j : Z_0 \rightarrow \mathbb{G}(k, N) \times \mathbb{G}(N-k, N)$  is the open embedding. Moreover, the weight filtration on  $\mathbb{G}(j_*\delta_{Z_0,m})\langle d_k \rangle$  agrees with the filtration on  $\mathcal{T}$  coming from Rickard's complex.*

**Remark 8.5.** Here Saito's pushforward of Hodge modules is very important in the statement of the theorem. In particular,  $\mathbb{G}(j_*\delta_{Z_0,m})$  and  $\int_j \delta_{Z_0,h}$  are *not* isomorphic as  $\mathcal{D}_h$ -modules.

By Proposition 5.2, we immediately deduce the following corollary which is used in this work of Bezrukavnikov-Losev [BL].

**Corollary 8.6.** *We have  $\mathcal{T} \otimes_{\mathbb{C}[h]} \mathbb{C}_1 \cong \int_j \mathcal{O}_{Z_0}[d_k]$ .*

**8.1. Filtrations and iterated cones.** The main step in proving Theorem 8.4 is showing that the associated graded pieces in the weight filtration of  $\mathbb{G}(j_*\delta_{Z_0,m})$  are the same as the terms in Rickard's complex used to define  $\mathcal{T}$  as an iterated cone. To do this we first recall a few generalities regarding filtrations and iterated cones.

Consider an object  $\mathcal{A}$  equipped with an increasing filtration

$$0 \subset W_0\mathcal{A} \subset W_1\mathcal{A} \subset \cdots \subset W_k\mathcal{A} = \mathcal{A}$$

where we denote the subquotients by  $\text{gr}_i\mathcal{A} := W_i\mathcal{A}/W_{i-1}\mathcal{A}$ . If  $k = 1$  then we have the exact triangle

$$\text{gr}_0\mathcal{A} \rightarrow \mathcal{A} \rightarrow \text{gr}_1\mathcal{A}$$

where  $\text{gr}_0\mathcal{A} = W_0\mathcal{A}$ . We can rewrite this triangle as  $\mathcal{A} \cong \text{Cone}(\text{gr}_1\mathcal{A}[-1] \rightarrow \text{gr}_0\mathcal{A})$ . If  $k > 1$  then we can repeat this argument to find that

$$(11) \quad \mathcal{A} \cong \text{Cone}(\text{gr}_k\mathcal{A}[-k] \rightarrow \text{gr}_{k-1}\mathcal{A}[-k+1] \rightarrow \cdots \rightarrow \text{gr}_1\mathcal{A}[-1] \rightarrow \text{gr}_0\mathcal{A}).$$

where the expression on the right is an iterated cone. Iterated cones, also called convolutions, were introduced by Orlov [O]. The definition is quite general and applies to any complex

$$A_k \xrightarrow{f_k} A_{k-1} \rightarrow \cdots \rightarrow A_1 \xrightarrow{f_1} A_0$$

of objects in a triangulated category. The convolution of a complex  $(A_\bullet, f_\bullet)$  may not exist and may not be unique. However, there are simple homological conditions under which both existence are assured.

**Lemma 8.7.** [CK1, Prop.8.3] *Let  $(A_\bullet, f_\bullet)$  be a complex in an abstract triangulated category.*

(i) *If  $\text{Hom}(A_{i+j+1}[j], A_i) = 0$  for all  $i \geq 0, j \geq 1$ , then any two convolutions of  $(A_\bullet, f_\bullet)$  are isomorphic.*

(ii) *If  $\text{Hom}(A_{i+j+2}[j], A_i) = 0$  for all  $i \geq 0, j \geq 1$ , then  $(A_\bullet, f_\bullet)$  has a convolution.*

**8.2. The associated graded of the weight filtration.** Consider the weight filtration  $W$  on  $j_*\delta_{Z_0, m}$  and the associated graded

$$\text{gr}_s^W(j_*\delta_{Z_0, m}) := W_s(j_*\delta_{Z_0, m})/W_{s-1}(j_*\delta_{Z_0, m}).$$

**Proposition 8.8.** *We have  $G(\bigoplus_s \text{gr}_s^W(j_*\delta_{Z_0, m})) \otimes_{\mathbb{C}[h]} \mathbb{C}_1 \cong \bigoplus_s \text{IC}_{\overline{Z}_s}$ .*

*Proof.* Each  $\text{gr}_s^W(j_*\delta_{Z_0, m})$  is a polarizable pure Hodge module of weight  $s$ . Therefore,

$$G(\text{gr}_s^W(j_*\delta_{Z_0, m})) \otimes_{\mathbb{C}[h]} \mathbb{C}_1$$

is isomorphic to a direct sum of IC  $\mathcal{D}$ -modules (c.f. [PS], Theorem 14.37). Because everything is  $GL_N$ -equivariant, these must be the IC objects of  $GL_N$  orbits. As all these orbits are simply connected, the only local systems arising must be trivial. Thus we obtain

$$(12) \quad G\left(\bigoplus_s \text{gr}_s^W(j_*\delta_{Z_0, m})\right) \otimes_{\mathbb{C}[h]} \mathbb{C}_1 \cong \bigoplus_s \text{IC}_{\overline{Z}_s}^{\oplus f_s}$$

for some  $f_s \in \mathbb{N}$ . It remains to show that all  $f_s = 1$ .

Fix  $0 \leq \ell \leq k$  and pick a point  $x_\ell \in Z_\ell$  and let  $i_\ell : \{x_\ell\} \rightarrow \mathbb{G}(k, N) \times \mathbb{G}(N - k, N)$  denote the inclusion of this point. Using base change for  $\mathcal{D}$ -modules and Proposition 5.2 we have

$$(13) \quad i_\ell^\dagger(G(j_*\delta_{Z_0, m}) \otimes_{\mathbb{C}[h]} \mathbb{C}_1) \cong i_\ell^\dagger \int_j \delta_{Z_0} \cong \begin{cases} \mathbb{C}[-2d_k] & \text{if } \ell = 0 \\ 0 & \text{otherwise} \end{cases}$$

On the other hand, applying  $i_\ell^\dagger$  to both sides of (12), we obtain

$$(14) \quad i_\ell^\dagger \left( \bigoplus_s G(\text{gr}_s^W(j_*\delta_{Z_0, m})) \otimes_{\mathbb{C}[h]} \mathbb{C}_1 \right) \cong \bigoplus_s i_\ell^\dagger \text{IC}_{\overline{Z}_s}^{\oplus f_s} \\ \cong \bigoplus_s \bigoplus_{\begin{bmatrix} \ell \\ s \end{bmatrix}} \mathbb{C}^{\oplus f_s}[s\ell - 2d_k]$$

where the second isomorphism follows from Lemma 8.10 (restricted to  $h = 1$ ). Taking Euler characteristics gives

$$\sum_s (-1)^{s\ell} f_s (-1)^{s(\ell-s)} \binom{\ell}{s} = \begin{cases} 1 & \text{if } \ell = 0 \\ 0 & \text{otherwise} \end{cases}$$

The upper-triangularity of this system of relations above means that there is at most one set of solutions  $f_s$ . Since

$$\sum_s (-1)^s \binom{\ell}{s} = \begin{cases} 1 & \text{if } \ell = 0 \\ 0 & \text{otherwise} \end{cases}$$

it follows that  $f_s = 1$ .  $\square$

**Corollary 8.9.** *For each  $s = 0, \dots, k$ , we have  $\mathrm{gr}_s^W(j_*\delta_{Z_0,m}) \cong \mathrm{IC}_{\overline{Z}_s,m}\{s\}$  and for  $p > s$ ,  $\mathrm{gr}_p^W(j_*\delta_{Z_0,m}) = 0$ .*

*Proof.* By Saito's results each  $\mathrm{gr}_p^W(j_*\delta_{Z_0,m})$  is a pure polarizable Hodge module of weight  $p$ . By the structure theorem, we have

$$\oplus_p \mathrm{gr}_s^W(j_*\delta_{Z_0,m}) = \oplus_t M_{\overline{Z}_t}$$

where  $M_{\overline{Z}_t}$  has strict support on  $\overline{Z}_t$ . On the other hand, from Proposition 8.8, we know that the underlying  $\mathcal{D}$ -module of  $\oplus_s \mathrm{gr}_s^W(j_*\delta_{Z_0,m})$  is a direct sum of IC objects one for each stratum. Thus if we restrict to an open subset of  $\overline{Z}_0$ , we see that  $M_{\overline{Z}_0}$  must restrict to a constant rank 1 variation of Hodge structure (since the direct sum of the IC  $\mathcal{D}$ -modules restricts to a trivial rank 1 flat connection). Thus  $M_{\overline{Z}_0} = \mathrm{IC}_{\overline{Z}_0,m}\{e_0\}$  for some integer  $e_0$ . We then remove this direct summand and continue. In this way, we conclude that  $M_{\overline{Z}_s} = \mathrm{IC}_{\overline{Z}_s,m}\{e_s\}$  for some integers  $e_s$ .

To determine the integers  $e_s$ , we just repeat the computation from the proof of Proposition 8.8 except working in the category of mixed Hodge modules. This gives us the equation

$$\sum_s (-1)^{e_s} q^{s\ell - 2d_k - e_s} \begin{bmatrix} \ell \\ s \end{bmatrix} = \begin{cases} q^{-2d_k} & \text{if } \ell = 0 \\ 0 & \text{otherwise} \end{cases}$$

The identity

$$\sum_s (-1)^s q^{s\ell - s} \begin{bmatrix} \ell \\ s \end{bmatrix} = \begin{cases} 1 & \text{if } \ell = 0 \\ 0 & \text{otherwise} \end{cases}$$

thus implies that  $e_s = s$ .

Hence we see that

$$\oplus_p \mathrm{gr}_p^W(j_*\delta_{Z_0,m}) = \oplus_{s=0}^k \mathrm{IC}_{\overline{Z}_s,m}\{s\}$$

Since for each  $p$ , there is at most one summand of weight  $p$  on the right hand side, the result follows.  $\square$

**Lemma 8.10.** *We have  $i_\ell^! \mathrm{IC}_{\overline{Z}_s,m} \cong \bigoplus_{\begin{bmatrix} \ell \\ s \end{bmatrix}} \mathbb{C}\langle s\ell - 2d_k \rangle$ .*

*Proof.* Recall the variety

$$P_s = \{(V'', V, V') \in \mathbb{G}(s, N) \times \mathbb{G}(k, N) \times \mathbb{G}(N - k, N) : V'' \subset (V \cap V')\}.$$

The natural map  $P_s \rightarrow \mathbb{G}(k, N) \times \mathbb{G}(N - k, N)$  has image  $\overline{Z}_s$  and by Proposition 6.2 the map  $\pi_s : P_s \rightarrow \overline{Z}_s$  is a small resolution. This means that  $\mathrm{IC}_{\overline{Z}_s,m} = \pi_{s*} \delta_{P_s,m}$ . Applying base change we get  $i_\ell^! \mathrm{IC}_{\overline{Z}_s,m} = \pi_{s*} \delta_{\pi_s^{-1}(x_\ell),m} \langle \dim \pi_s^{-1}(x_\ell) - \dim P_s \rangle$ . Note that

$$\dim(\pi_s^{-1}(x_\ell)) - \dim(P_s) = s(\ell - s) - (2d_k - s^2) = s\ell - 2d_k.$$

Now  $\pi_s^{-1}(x_\ell) \cong \mathbb{G}(s, \ell)$  and thus

$$i_\ell^! \mathrm{IC}_{\overline{Z}_s,m} \cong H^*(\mathbb{G}(s, \ell)) \langle s\ell - 2d_k \rangle \cong \bigoplus_{\begin{bmatrix} \ell \\ s \end{bmatrix}} \mathbb{C}\langle s\ell - 2d_k \rangle.$$

□

### 8.3. Proof of Theorem 8.4.

**Proposition 8.11.** *We have  $\Theta^s \cong \mathrm{IC}_{\overline{Z}_s, h} \langle d_k \rangle$ .*

*Proof.* Applying similar reasoning as in the proof of Proposition 6.3 we see that

$$\mathcal{F}^{(N-k-s)} * \mathcal{E}^{(k-s)} = \int_{\pi} \delta_{P_s, h} \langle d_k \rangle.$$

By Proposition 6.2, we know that  $\int_{\pi} \delta_{P_s, h} = \mathrm{IC}_{\overline{Z}_s, h}$ . The result follows. □

We will now complete the proof of Theorem 8.4. Recall that  $\mathcal{T}$  is the right convolution of  $\Theta^k \langle -k \rangle \rightarrow \cdots \rightarrow \Theta^0$ . By Proposition 8.11, we can rewrite this complex (up to an overall  $\langle d_k \rangle$ ) as

$$\mathrm{IC}_{\overline{Z}_k, h} \langle -k \rangle \rightarrow \cdots \rightarrow \mathrm{IC}_{\overline{Z}_1, h} \langle -1 \rangle \rightarrow \mathrm{IC}_{\overline{Z}_0, h}$$

By Lemma 8.12 below, we know that the differentials in this complex are unique (up to scalar).

Denote  $U_s := Z_0 \cup Z_1 \cdots \cup Z_s$  and let  $j_s : Z_0 \rightarrow U_s$  be the natural inclusion (note that  $j_k = j$ ). We will show by induction that  $\mathbf{G}(j_{s*} \delta_{Z_0, m})$  is the iterated cone of a complex

$$(15) \quad \mathrm{IC}_{\overline{Z}_s, h} \langle -s \rangle \rightarrow \cdots \rightarrow \mathrm{IC}_{\overline{Z}_1, h} \langle -1 \rangle \rightarrow \mathrm{IC}_{\overline{Z}_0, h}$$

restricted to  $U_s$ . The base case is trivial. By Corollary 8.9 we know that

$$\bigoplus_i \mathbf{G}(\mathrm{gr}_i^W(j_{s*} \delta_{Z_0, m})) \cong \bigoplus_{i=1}^s \mathrm{IC}_{\overline{Z}_s, h} \{s\}.$$

Moreover, the weight filtration of  $j_{s*} \delta_{Z_0, m}$  restricted to  $U_{s-1}$  agrees with the weight filtration of  $j_{(s-1)*} \delta_{Z_0, m}$ . These two facts together with the induction hypothesis imply that  $\mathbf{G}(j_{s*} \mathcal{O}_{Z_0, m})$  is isomorphic to a complex of the form

$$\mathrm{IC}_{\overline{Z}_{s-1}, h} \langle -s+1 \rangle \rightarrow \cdots \rightarrow \mathrm{IC}_{\overline{Z}_i, h} \langle -i \rangle \oplus \mathrm{IC}_{\overline{Z}_s, h} [-i] \{s\} \rightarrow \cdots \rightarrow \mathrm{IC}_{\overline{Z}_0, h}$$

restricted to  $U_s$ , for some  $i$ . However, if  $i \neq s$  then by Lemma 8.12 all the maps to or from  $\mathrm{IC}_{\overline{Z}_s, h} [-i] \{s\}$  would be  $\mathbb{C}[h]$ -torsion and hence  $\mathbf{G}(j_{s*} \delta_{Z_0, m}) \otimes_{\mathbb{C}[h]} \mathbb{C}_1 \cong j_{s*} \mathcal{O}_{Z_0}$  would be decomposable (contradiction). It follows that  $i = s$  and hence  $\mathbf{G}(j_{s*} \delta_{Z_0, m})$  is of the form (15).

**Lemma 8.12.** *We have that*

- $\mathrm{Hom}_{\mathbb{C}[h]}(\mathrm{IC}_{\overline{Z}_j, h} \{j\} [-l], \mathrm{IC}_{\overline{Z}_i, h} \{i\})$  is  $\mathbb{C}[h]$ -torsion if  $l \neq j - i$  and
- $\mathrm{Hom}_{\mathbb{C}[h]}(\mathrm{IC}_{\overline{Z}_{i+1}, h} \langle -1 \rangle, \mathrm{IC}_{\overline{Z}_i, h}) \cong \mathbb{C}[h]$ .

*Proof.* Using Proposition 8.11 we can identify  $\mathrm{IC}_{\overline{Z}_i, h}$  with  $\mathcal{F}^{(N-k-i)} * \mathcal{E}^{(k-i)}$  (up to a shift by  $\langle d_k \rangle$  which does not depend on  $i$  and we will ignore). On the other hand, by applying adjunction and the  $\mathfrak{sl}_2$  commutator relation repeatedly one can decompose

$$(16) \quad \mathrm{Hom}_{\mathbb{C}[h]}(\mathcal{F}^{(N-k-j)} * \mathcal{E}^{(k-j)} \{j\} [-l], \mathcal{F}^{(N-k-i)} * \mathcal{E}^{(k-i)} \{i\})$$

as a direct sum of terms of the form  $\mathrm{Hom}_{\mathbb{C}[h]}(\mathrm{Id}_{\mu}, \mathrm{Id}_{\mu}[l] \{i-j\} \langle a \rangle)$  for various  $\mu$  and  $a \in \mathbb{Z}$ . On the other hand

$$\mathrm{Hom}_{\mathbb{C}[h]}^*(\mathrm{Id}_{\mu}, \mathrm{Id}_{\mu}) = \mathrm{Hom}_{\mathbb{C}[h]}^* \left( \int_{\Delta} \delta_{\mathbf{G}(k, N), h}, \int_{\Delta} \delta_{\mathbf{G}(k, N), h} \right)$$

which, using adjunction, is (modulo  $\mathbb{C}[h]$ -torsion) isomorphic to a direct sum of terms of the form  $\mathbb{C}[h]\langle a \rangle$ . Since  $\langle a \rangle = [a]\{-a\}$ , (16) is  $\mathbb{C}[h]$ -torsion unless  $l + i - j = 0$ . This proves the first assertion.

The second assertion is proved similarly by using the  $\mathfrak{sl}_2$  commutator relation to simplify the Hom space (see for instance [CKL1]). In the end we end up with a direct sum of terms of the form  $\mathrm{Hom}_{\mathbb{C}[h]}^*(\mathrm{Id}_\mu, \mathrm{Id}_\mu)$  where  $* \leq 0$  (with only one term where  $* = 0$ ). The result then follows since

$$\mathrm{Hom}_{\mathbb{C}[h]}^*(\mathrm{Id}_\mu, \mathrm{Id}_\mu) \cong \begin{cases} \mathbb{C}[h] & \text{if } * = 0 \\ 0 & \text{if } * < 0 \end{cases}$$

□

**Remark 8.13.** It is possible to show directly that  $j_*\delta_{Z_0,m}$  is a kernel which induces an equivalence without relating it to a categorical  $\mathfrak{sl}_2$  action. This fact is well-known to experts though we were not able to find a proof in the literature. On the other hand, this still does not immediately imply that  $G(j_*\delta_{Z_0,m})$  induces an equivalence between categories of  $\mathcal{D}_h$ -modules.

**Remark 8.14.** Instead of  $j_*\delta_{Z_0,m}$  we can equally well have considered  $j!\delta_{Z_0,m}$ . In this case one can show that  $\mathcal{T}^{-1} \cong G(j!\delta_{Z_0,m})\langle d_k \rangle$ .

## 9. ASSOCIATED GRADED OF T

In Theorem 8.4 we saw that the equivalence  $T$  is induced by  $\mathcal{T} \cong G(j_*\mathcal{O}_{Z_0,m})\langle d_k \rangle$ . The associated graded  $\mathrm{gr}(G(j_*\mathcal{O}_{Z_0,m}))$  can be described as the convolution of the complex  $\mathrm{gr}(\Theta^*)$  or equivalently, coming from the categorical  $\mathfrak{sl}_2$  action defined in Corollary 7.2.

More precisely, recall from [C1] the locus inside  $T^*\mathbb{G}(k, N) \times T^*\mathbb{G}(N - k, N)$  given by

$$\mathfrak{Z}(k, N) := \{(X, V, V') : \dim(V) = k, \dim(V') = N - k, X(\mathbb{C}^N) \subset V \cap V', X(V) = X(V') = 0\}$$

where  $V, V' \subset \mathbb{C}^N$  and  $X \in \mathrm{End}(\mathbb{C}^N)$ . This is just the fiber product of  $T^*\mathbb{G}(k, N)$  and  $T^*\mathbb{G}(N - k, N)$  over their common affinization. This variety consists of  $k + 1$  irreducible components  $\mathfrak{Z}_s(k, N)$  where  $s = 0, \dots, k$  and  $\mathfrak{Z}_s(k, N)$  is the locus where

$$\dim(\ker X) \geq N - s \quad \text{and} \quad \dim(V \cap V') \geq s.$$

Note that  $\mathfrak{Z}_s$  is the closure of the conormal bundle to  $Z_s$ .

**Proposition 9.1.** *For each  $\Theta^s$  we have*

$$\mathrm{gr}(\Theta^s) \cong \tilde{\mathcal{O}}_{\mathfrak{Z}_s(k, N)} \otimes \det(V)^{-s} \otimes \det(V')^{N-s} \otimes \det(\mathbb{C}^N)^{-N+k+s} \{d_k - s^2\}$$

where  $\tilde{\mathcal{O}}_{\mathfrak{Z}_s(k, N)}$  denotes the normalized structure sheaf.

**Remark 9.2.** We expect that  $\mathfrak{Z}_s(k, N)$  is normal in which case  $\tilde{\mathcal{O}}_{\mathfrak{Z}_s(k, N)} = \mathcal{O}_{\mathfrak{Z}_s(k, N)}$ .

*Proof.* By Proposition 4.2 we have

$$\mathrm{gr}(\mathcal{F}^{(N-k-s)} * \mathcal{E}^{(k-s)}) \cong \mathrm{gr}(\mathcal{F}^{(N-k-s)}) * \mathrm{gr}(\mathcal{E}^{(k-s)}).$$

Both these terms are identified in Corollary 7.2 and they agree with the categorical  $\mathfrak{sl}_2$  action from [CK4]. The result now follows from Proposition A.7 of [CK4] keeping in mind that  $\mathfrak{Z}_s(k, N)$  corresponds to  $Z_{k-s}(k, N - k)$  in the notation from [CK4]. □

Thus,  $\vec{\text{gr}}(\mathcal{T})$  is the convolution of the complex made up of  $\vec{\text{gr}}(\Theta^s)$  which are described in Proposition 9.1. Although this description of  $\vec{\text{gr}}(\mathcal{T})$  is quite useful to work with we now give another description which more closely resembles the definition of  $\mathcal{T}$  as the pushforward  $j_*\mathcal{O}_{Z_{0,m}}$ . It would be interesting to relate directly these two descriptions of  $\mathcal{T}$  and  $\vec{\text{gr}}(\mathcal{T})$ .

We begin by defining open subschemes  $\mathfrak{Z}_s^o(k, N) \subset \mathfrak{Z}_s(k, N)$  by the condition

$$\dim(\ker X) + \dim(V \cap V') \leq N + 1.$$

We denote by  $\mathfrak{Z}^o(k, N) \subset \mathfrak{Z}(k, N)$  the union of all  $\mathfrak{Z}_s^o(k, N)$  and by  $f : \mathfrak{Z}^o \hookrightarrow \mathfrak{Z}$  their inclusion. The advantage of  $\mathfrak{Z}^o$  is that it avoids the more complicated singularities of  $\mathfrak{Z}$  but is big enough that the complement  $\mathfrak{Z} \setminus \mathfrak{Z}^o$  has codimension at least two.

**Lemma 9.3.** [C1, Lemma 3.4] *The intersection  $D_{s,+}^o(k, N) := \mathfrak{Z}_s^o(k, N) \cap \mathfrak{Z}_{s+1}(k, N)$  is a Cartier divisor in  $\mathfrak{Z}_s^o(k, N)$  and corresponds to the locus where  $\dim \ker(X) = N - s$  and  $\dim(V \cap V') = s + 1$ .*

**Proposition 9.4.** *On  $\mathfrak{Z}^o(k, N)$  there exists a line bundle  $\mathcal{L}(k, N)$  uniquely determined by its restriction*

$$\mathcal{L}(k, N)|_{\mathfrak{Z}_s^o(k, N)} \cong \mathcal{O}_{\mathfrak{Z}_s^o(k, N)}([D_{s,+}^o(k, N)]) \otimes \det(V)^{-s} \otimes \det(V')^{N-s} \otimes \det(\mathbb{C}^N)^{-N+k+s} \{-s(s-1)\}$$

to each component of  $\mathfrak{Z}^o(k, N)$ . Moreover,  $\vec{\text{gr}}(\mathcal{T}) \cong R^0 f_*(\mathcal{L}(k, N))\{d_k\}$ .

*Proof.* This result follows from Proposition A.8 of [CK4] with a minor amount of work.  $\square$

**Corollary 9.5.** *We have  $\text{gr}(\mathcal{G}(j_*\mathcal{O}_{Z_{0,m}})) \cong R^0 f_*(\mathcal{L}'(k, N))$  where  $\mathcal{L}'(k, N)$  is determined by its restrictions*

$$\mathcal{L}'(k, N)|_{\mathfrak{Z}_s^o(k, N)} \cong \mathcal{O}_{\mathfrak{Z}_s^o(k, N)}([D_{s,+}^o(k, N)]) \otimes \det(V)^{-s} \otimes \det(\mathbb{C}^N/V')^s \{-s(s-1)\}.$$

*Proof.* This follows from Theorem 8.4 and the last assertion of the previous proposition once we unpack the definitions of  $\vec{\text{gr}}$  and  $\mathcal{T}$ . In particular,  $\mathcal{L}'(k, N) = \mathcal{L}(k, N) \otimes f^*\omega_{\mathbb{G}(N-k, N)}^\vee$  which gives the expression above for  $\mathcal{L}'(k, N)$ .  $\square$

**Remark 9.6.** It is probably worthwhile recalling the reason behind this result. Consider the union of two equi-dimensional varieties  $Y := Y_0 \cup Y_1$  along a divisor  $D := Y_0 \cap Y_1$ . Then the standard short exact sequence

$$0 \rightarrow \mathcal{O}_{Y_0}(-D) \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_{Y_1} \rightarrow 0$$

can be rewritten as  $\mathcal{O}_Y \cong \text{Cone}(\mathcal{O}_{Y_1}[-1] \rightarrow \mathcal{O}_{Y_0}(-D))$ . More generally, if  $Y = Y_0 \cup \dots \cup Y_k$  so that  $Y_s \cap Y_{s+1}$  is a divisor and  $Y_s \cap Y_{s'} = \emptyset$  if  $|s - s'| > 1$  then one can recover  $\mathcal{O}_Y$  as a convolution

$$\mathcal{O}_Y \cong \text{Cone}(\mathcal{O}_{Y_k}[-k] \rightarrow \dots \rightarrow \mathcal{O}_{Y_1}(-D_1)[-1] \rightarrow \mathcal{O}_{Y_0}(-D_0))$$

for some divisors  $D_s$ . In our case above, we are building a sheaf on  $\mathfrak{Z}^o(k, N)$  by glueing together appropriate line bundles on each  $\mathfrak{Z}_s^o(k, N)$ . To finish, a formal argument shows that  $\text{gr}(j_*\mathcal{O}_U)$  must be Cohen-Macaulay and, in particular,  $S_2$ . This means that it is uniquely determined by its restriction to an open subset whose complement has codimension  $\geq 2$ . This restriction can be identified with  $\mathcal{L}(k, N)$  which subsequently implies Corollary 9.5.

## 10. GENERALIZATION TO COMINUSCULE FLAG VARIETIES

In this section, we outline a possible generalization of our results to cominuscule flag varieties. Other than the existence of a categorical  $\mathfrak{sl}_2$ -action, we expect that the results of this paper generalize nicely to this setting.

**10.1. Geometric setup.** Let  $G$  be a simple complex algebraic group. Recall that a node  $i$  of the Dynkin diagram of  $G$  is called cominuscule if, for every positive root  $\alpha$ , the coefficient of  $\alpha_i$  in  $\alpha$  is at most 1. Let  $i$  be a cominuscule node and let  $P$  be the associated maximal parabolic subgroup. In this case,  $G/P$  is called a cominuscule flag variety. Let  $Q = P^T$  be the opposite parabolic subgroup (i.e.  $\mathfrak{g}_\beta \subset \mathfrak{q}$  iff.  $\mathfrak{g}_{-\beta} \subset \mathfrak{p}$ ).  $G/Q$  is also a cominuscule flag variety.

Denote by  $W$  the Weyl group of  $G$  and let

$$A = \{w \in W : w < s_j w \text{ and } w < w s_j, \text{ for all } j \neq i\}.$$

Sometimes these are called biGrassmannian elements of  $W$ .  $A$  carries a partial order by restricting the Bruhat order of  $W$ . A standard result tells us that there is an order-reversing bijection between  $A$  and the set of  $G$ -orbits on  $G/P \times G/Q$ . Also for any element  $w$  of  $A$  the corresponding orbit has codimension  $\ell(w)$ .

We have checked the following lemma in a case-by-case fashion. We believe that there should be a general proof.

**Lemma 10.1.** *The set  $A$  is linearly ordered.*

Let  $k + 1$  be the cardinality of  $A$  and let  $Z_0, \dots, Z_k$  be the  $G$  orbits on  $G/P \times G/Q$ , in decreasing size, so  $Z_0$  denotes the open orbit.

## 10.2. Some examples.

**10.2.1. Usual Grassmannians.** If we take  $G = SL_N$ , then we can choose any node  $i$  of the diagram and we get  $G/P = \mathbb{G}(i, N)$ ,  $G/Q = \mathbb{G}(N - i, N)$  and we return to our previous setup (in this case,  $k = \min(i, N - i)$ ).

**10.2.2. Even dimensional quadric.** If we take  $G = SO_{2n}$  and  $i = 1$ , then  $G/P = \{[x_1, \dots, x_{2n}] : x_1^2 + \dots + x_{2n}^2 = 0\}$  is a quadric in  $\mathbb{P}^{2n-1}$ . In this case  $G/Q = G/P$  and  $k = 2$ . Moreover, we have a simple description of the  $G$ -orbits.

$$Z_0 = \{([x], [y]) : x \cdot y \neq 0\}, \quad Z_1 = \{([x], [y]) : x \cdot y = 0, x \neq y\}, \quad Z_2 = \{([x], [y]) : x = y\}.$$

**10.2.3. Lagrangian Grassmannian.** Assume that  $n$  is odd. If we take  $G = SO_{2n}$  and  $i = n$ , then  $G/P = OG(n, 2n)_+$ , one connected component of the variety of Lagrangian subspaces of  $\mathbb{C}^{2n}$  (with respect to the symmetric bilinear form). In this case,  $G/Q = OG(n, 2n)_-$ , the other connected component. In this case  $k = \frac{n-1}{2}$  and

$$Z_0 = \{(V, W) : \dim(V \cap W) = 0\}, \quad Z_1 = \{(V, W) : \dim(V \cap W) = 2\}, \\ \dots, \quad Z_k = \{(V, W) : \dim(V \cap W) = n - 1\}.$$

Note that for  $V \in OG(n, 2n)_+$ ,  $W \in OG(n, 2n)_-$ ,  $\dim(V \cap W)$  is always even.



**10.3. The  $\mathcal{D}$ -module equivalence.** Let  $j : Z_0 \rightarrow G/P \times G/Q$  be the inclusion. Note that  $\overline{Z_1}$  is a divisor in  $G/P \times G/Q$ , since it is associated to the element  $s_i \in A$  which has length 1. Thus  $j$  is an affine morphism. Thus,  $\int_j \mathcal{O}_{Z_0}$  is a  $\mathcal{D}_{G/P \times G/Q}$ -module (there is no higher direct image). The following result should not be difficult to establish.

**Conjecture 10.2.**  $\int_j \delta_{Z_0}$  is the kernel for an equivalence  $D(\mathcal{D}_{G/P}\text{-mod}) \xrightarrow{\sim} D(\mathcal{D}_{G/Q}\text{-mod})$ .

We may also consider the Hodge module push-forward  $j_* \delta_{Z_0, m}$  and its underlying  $\mathcal{D}_h$ -module  $\mathcal{T} = G(j_* \delta_{Z_0, m}) \langle \dim(G/P) \rangle$ . We can extend the previous conjecture as follows.

**Conjecture 10.3.**  $\mathcal{T}$  is the kernel for an equivalence  $D(\mathcal{D}_{G/P, h}\text{-mod}) \xrightarrow{\sim} D(\mathcal{D}_{G/Q, h}\text{-mod})$ .

As in section 8.2, we can consider the weight filtration. We hope that similar techniques as in that section will help to establish the following result.

**Conjecture 10.4.** For  $s = 0, \dots, k$ , we have  $gr_s^W(j_* \delta_{Z_0, m}) = \text{IC}_{\overline{Z}_s, m} \{s\}$  and the rest of the subquotients are zero.

**10.4.  $\mathcal{O}$ -module side.** Now, we consider kernels on the product of the cotangent bundles  $T^*G/P \times T^*G/Q$ . For each  $s$ , let  $\mathfrak{Z}_s = T_{Z_s}^*(G/P \times G/Q) \subset T^*G/P \times T^*G/Q$  be the closure of the conormal bundle of  $Z_s$ . These components  $\mathfrak{Z}_0, \dots, \mathfrak{Z}_k$  are the irreducible components of  $\mathfrak{Z} = T^*G/P \times_{\mathfrak{g}} T^*G/Q$ .

Let  $Q_s = \text{gr}(\text{IC}_{\overline{Z}_s, h} \langle \dim(G/P) \rangle \{s\})$ . If we assume the conjectures of the previous section, then we obtain the following.

**Corollary 10.5.**  $\text{gr}(\mathcal{T})$  is the kernel for an equivalence  $D(\mathcal{O}_{T^*G/P}\text{-mod}) \rightarrow D(\mathcal{O}_{T^*G/Q}\text{-mod})$ . Moreover, there is a filtration of  $\text{gr}(\mathcal{T})$  whose subquotients are the sheaves  $Q_s$ .

We can try to give a more intrinsic construction of  $\text{gr}(\mathcal{T})$  as follows. First, we have the following conjectural description of the sheaves  $Q_s$ .

**Conjecture 10.6.**  $Q_s$  is supported on  $\mathfrak{Z}_s$ . In fact it is the pushforward of a line bundle on the normalization of  $\mathfrak{Z}_s$ .

As in the  $T^*\mathbb{G}(k, N)$  case, we hope that this will lead to a nice description of  $\text{gr}(\mathcal{T})$ .

**Conjecture 10.7.** There exists an dense subset  $\mathfrak{Z}^o$  of  $\mathfrak{Z}$  and a line bundle  $\mathcal{L}$  on  $\mathfrak{Z}^o$ , such that  $\text{gr}(\mathcal{T}) \cong R^0 f_* \mathcal{L}$  (where  $f : \mathfrak{Z}^o \rightarrow \mathfrak{Z}$  is the inclusion).

Of course, it would be nice to explicitly describe this open subset and this line bundle.

## REFERENCES

- [BLM] A. Beilinson, G. Lusztig and R. MacPherson, A geometric setting for the quantum deformation of  $GL_n$ , *Duke Math. J.* **61** (1990), no. 2, 655–677.
- [BL] R. Bezrukavnikov and I. Losev, Etingof conjecture for quantized quiver varieties, [arXiv:1309.1716](#).
- [C1] S. Cautis, Equivalences and stratified flops, *Compositio Math.* **148** (2012), no. 1, 185–209; [arXiv:0909.0817](#).
- [C2] S. Cautis, Rigidity in higher representation theory; [arXiv:1409.0827](#).
- [CK1] S. Cautis and J. Kamnitzer, Knot homology via derived categories of coherent sheaves I,  $\mathfrak{sl}_2$  case, *Duke Math. J.* **142** (2008), no. 3, 511–588; [math.AG/0701194](#).
- [CK2] S. Cautis and J. Kamnitzer, Knot homology via derived categories of coherent sheaves II, *Invent. Math.* **174** (2008), no. 1, 165–232; [arXiv:0710.3216](#).

- [CK3] S. Cautis and J. Kamnitzer, Braiding via geometric categorical Lie algebra actions, *Compositio Math.* **148** (2012), no. 2, 464–506; [arXiv:1001.0619](#).
- [CK4] S. Cautis and J. Kamnitzer, Knot homology via derived categories of coherent sheaves IV, coloured links; [arXiv:1410.7156](#).
- [CDK] S. Cautis, C. Dodd and J. Kamnitzer, Categorical actions on quiver varieties: from  $\mathcal{D}$ -modules to coherent sheaves (in preparation).
- [CKL1] S. Cautis, J. Kamnitzer and A. Licata, Derived equivalences for cotangent bundles of Grassmannians via categorical  $\mathfrak{sl}_2$  actions, *J. Reine Angew. Math.* **675** (2013), 53–99; [arXiv:0902.1797](#).
- [CKL2] S. Cautis, J. Kamnitzer and A. Licata, Coherent sheaves on quiver varieties and categorification, *Math. Ann.* **357** (2013), no. 3, 805–854; [arXiv:1104.0352](#).
- [CL] S. Cautis and A. Lauda, Implicit structure in 2-representations of quantum groups, *Selecta Math.* **21** (2015), no. 1, 201–244; [arXiv:1111.1431](#).
- [CR] J. Chuang and R. Rouquier, Derived equivalences for symmetric groups and  $\mathfrak{sl}_2$ -categorification; *Annals of Mathematics*, **167** (2008), 245–298. [math.RT/0407205](#).
- [G] D. Gaitsgory, Functors given by kernels, adjunctions and duality; [arXiv:1303.2763](#).
- [HTT] R. Hotta, K. Takeuchi, and T. Tanisaki,  $\mathcal{D}$ -modules, perverse sheaves, and representation theory, *Progress in Mathematics*, **236**, Birkhauser Boston, 2008.
- [Ld] A. Lauda, A categorification of quantum  $\mathfrak{sl}_2$ , *Adv. in Math.*, Vol. 225, Issue 6 (2010), 3327–3424; [arXiv:0803.3652v2](#).
- [Lm] G. Laumon, Sur la catégorie dérivée des  $\mathcal{D}$ -modules filtrés, *Lecture Notes in Math.* **1016**, Springer, Berlin 1983, 151–237.
- [O] D. Orlov, Equivalences of derived categories and K3 surfaces, *J. Math. Sci. (New York)* **84** (1997), no. 5, 1361–1381.
- [R] R. Rouquier, 2-Kac-Moody algebras; [arXiv:0812.5023](#).
- [PS] Chris Peters and Joseph Steenbrink, Mixed Hodge structures, *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*, **52**, Springer-Verlag, Berlin, 2008.
- [Sa1] M. Saito, Modules de Hodge polarisables, *Publ. RIMS. Kyoto Univ.* **24** (1988), 849–995.
- [Sa2] M. Saito, Mixed Hodge Modules, *Publ. RIMS. Kyoto Univ.* **26** (1990), 221–333.
- [Sc] C. Schnell, An overview of Morihiko Saito’s theory of mixed Hodge modules; [arXiv:1405.3096](#).
- [W] B. Webster, A categorical action on quantized quiver varieties; [arXiv:1208.5957](#).
- [WW] B. Webster and G. Williamson, A geometric construction of colored HOMFLYPT homology; [arXiv:0905.0486](#).
- [Ze] A. Zelevinsky, Small resolutions of singularities of Schubert varieties, *Functional Anal. Appl.* **17** (1983), no. 2, 142–144.
- [Z1] H. Zheng, A geometric categorification of tensor products of  $U_q(\mathfrak{sl}_2)$ -modules; [arXiv:0705.2630](#).
- [Z2] H. Zheng, Categorification of integrable representations of quantum groups, *Acta Math. Sin.*, **30** (2014), no. 6, 899–932; [arXiv:0803.3668](#).

*E-mail address:* [cautis@math.ubc.ca](mailto:cautis@math.ubc.ca)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER BC, CANADA

*E-mail address:* [cdodd@perimeterinstitute.ca](mailto:cdodd@perimeterinstitute.ca)

PERIMETER INSTITUTE FOR THEORETICAL PHYSICS, WATERLOO ON, CANADA

*E-mail address:* [jkamnitz@math.utoronto.ca](mailto:jkamnitz@math.utoronto.ca)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO ON, CANADA